A NEW PROOF OF TANAKA'S THEOREM

ALI ENAYAT

Abstract. We present a new proof of a theorem of Kazuyuki Tanaka, which states that every countable nonstandard model of WKL_0 has a non-trivial self-embedding onto a proper initial segment of itself. Moreover, the new proof has an ingredient that yields a novel characterization of models of WKL_0 among countable models of RCA_0.

1. Introduction

Friedman [6, Theorem 4.4] unveiled the striking result that every countable nonstandard model of PA is isomorphic to a proper initial segment of itself. One of the earliest refinements of Friedman's theorem is due to Lessan [10], who showed that a countable model $\mathcal{M}$ of Π_2^{PA} (the Π_2-consequences of PA) is isomorphic to a proper initial segment of itself iff $\mathcal{M}$ is 1-tall and 1-extendable. Here '$\mathcal{M}$ is 1-tall' means that the set of Σ_1-definable elements of $\mathcal{M}$ is not cofinal in $\mathcal{M}$, and '$\mathcal{M}$ is 1-extendable' means that there is an end extension $\mathcal{M}^*$ of $\mathcal{M}$ that satisfies $\mathrm{I}\Delta_0$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{M}^*)$. Dimitracopoulos and Paris [4], in turn, strengthened Lessan's aforementioned result by weakening Π_2^{PA} to $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathsf{exp}$, and they used this strengthening to show that every countable nonstandard model of $\mathrm{I}\Sigma_1$ is isomorphic to a proper initial segment of itself. This strengthening of Friedman's theorem was independently established, through a different line of reasoning, by Ressayre [12] in the following stronger form indicated in the 'moreover clause' (see [8, Ch. IV, Sec. 2(d)] for a detailed exposition of Ressayre's proof).

1.1. Theorem. *Every countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ is isomorphic to a proper initial segment of itself. Moreover, for any prescribed $a \in M$, there is a proper initial segment I of $\mathcal{M}$, and an isomorphism $\phi_a : \mathcal{M} \to I$ such that $\phi_a(m) = m$ for all $m \leq a$.*[1]

The point of departure for this paper is a theorem of Tanaka [14] that extends Theorem 1.1 to models of the fragment WKL_0 of second order arithmetic. Models of WKL_0 are two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M} = (M, +, \cdot, <, 0, 1) \models \mathrm{I}\Sigma_1$, and $\mathcal{A}$ is a family of subsets of M such that $(\mathcal{M}, \mathcal{A})$ satisfies the following three conditions:

[1] Ressayre also noted that the existence of such a family of embeddings $\{\phi_a : a \in M\}$ characterizes models of $\mathrm{I}\Sigma_1$ among countable models of $\mathrm{I}\Delta_0$.

(1) Induction for Σ^0_1 formulae;
(2) Comprehension for Δ^0_1-formulae; and
(3) Weak König's Lemma: every infinite subtree of the full binary tree has an infinite branch.

It is well known that every countable model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ can be expanded to a model $(\mathcal{M}, \mathcal{A}) \models \mathsf{WKL}_0$. This important result is due independently to Harrington and Ratajczyk; see [13, Lemma IX.1.8 + Theorem IX.2.1]. Therefore Theorem 1.2 below is a strengthening of Theorem 1.1.

1.2. Theorem. (Tanaka) *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *is isomorphic to a proper initial segment I of itself in the sense that there is an isomorphism $\phi : \mathcal{M} \to I$ such that ϕ induces an isomorphism $\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{A} \restriction I)$. Moreover, given any prescribed $a \in M$, there is some I and ϕ as above such that $\phi_a(m) = m$ for all $m \leq a$.*

In Theorem 1.2, $\mathcal{A} \restriction I := \{A \cap I : A \in \mathcal{A}\}$, and the isomorphism $\widehat{\phi}$ induced by ϕ is defined by: $\widehat{\phi}(m) = \phi(m)$ for $m \in M$ and $\widehat{\phi}(A) = \{\phi(a) : a \in A\}$ for $A \in \mathcal{A}$. Tanaka's proof of Theorem 1.2 in [14] is based on an elaboration of Ressayre's proof of Theorem 1.1, which uses game-theoretic ideas.

Tanaka's motivation for his result, as pointed out in [14, Sec. 3], was the development of non-standard methods within the confines of the frugal system WKL_0. A remarkable application of Tanaka's result appears in the work of Tanaka and Yamazaki [15], where it is used to show that the construction of the Haar measure (over compact groups) can be implemented within WKL_0 via a detour through nonstandard models. This is in contrast to the previously known constructions of the Haar measure whose implementation can only be accommodated within the stronger fragment ACA_0 of second order arithmetic. As it turns out, the known applications of Tanaka's theorem in the development of nonstandard methods do not need the full force of Theorem 1.2, but rather they rely on the following immediate corollary of the first assertion of Theorem 1.2.

1.3. Corollary. *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *has an extension* $(\mathcal{M}^*, \mathcal{A}^*)$ *to a model of* WKL_0 *such that* $\mathcal{M}^*$ *properly end extends* $\mathcal{M}$, *and* $\mathcal{A} = \mathcal{A}^* \restriction M$.

In this paper we present a new proof of the first assertion of Tanaka's Theorem (in Section 3). As shown in Theorem 3.6 our work also yields a new characterization of models of WKL_0 among countable models of RCA_0. Before presenting the new proof, we first need to discuss a self-embedding theorem (in Section 2) that plays a key role in our new proof. As indicated in Remark 3.4, our method can be extended to also establish the 'moreover' clause of Tanaka's Theorem. Further applications of the methodology of our proof of Theorem 1.2 will be presented in [5].

2. The Solovay-Paris Self-Embedding Theorem

Paris [11, Theorem 4] showed that every countable recursively saturated model of $\mathrm{I}\Delta_0+\mathrm{B}\Sigma_1$ is isomorphic to a proper initial segment of itself, a result that is described by Paris as being 'implicit' in an (unpublished) paper of Solovay.[2] This result of Solovay and Paris can be fine-tuned as in Theorem 2.1 below, following the strategy (with $n = 0$) of [9, Theorem 12.3]. The details of the proof of Theorem 2.1 have been worked out by Cornaros in [3, Corollary 11] and Yokoyama [16, Lemma 1.3].

2.1. Theorem. *Suppose $\mathcal{N}$ is a countable model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ that is recursively saturated, and there are $a < b$ in $\mathcal{N}$ such that for every Δ_0-formula $\delta(x, y)$ we have:*

$$(*) \qquad \mathcal{N} \models \exists y\ \delta(a, y) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(a, y).$$

There is an isomorphism $\phi : \mathcal{N} \to I$, where I is an initial segment of $\mathcal{N}$ with $a < I < b$, and $\phi(a) = a$.

2.2. Remark. Condition $(*)$ of Theorem 2.1 can be rephrased as

$$f(a) < b \text{ for all partial } \mathcal{N}\text{-recursive functions } f,$$

with the understanding that a partial function f from N to N is a partial $\mathcal{N}$-recursive function iff the graph of f is definable in $\mathcal{N}$ by a parameter-free Σ_1-formula.

3. Proof of Tanaka's Theorem

The goal of this section is to show that every countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of WKL_0 is isomorphic to a proper initial segment of itself. In Remark 3.4 we will comment on how our method can be adapted so as to also establish the 'moreover clause' of Tanaka's Theorem. Our proof has three stages:

- *Stage 1:* Given a countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of WKL_0, and a prescribed $a \in M$ in this stage we use the 'muscles' of $\mathrm{I}\Sigma_1$ in the form of the strong Σ_1-collection [8, Theorem 2.23, p.68] to locate an element b in $\mathcal{M}$ such that $f(a) < b$ for all partial $\mathcal{M}$-recursive functions f (as defined in Remark 2.2).
- *Stage 2 Outline:* We build an *end extension* $\mathcal{N}$ of $\mathcal{M}$ such that the following conditions hold:

  (I) $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$;
  (II) $\mathcal{N}$ is recursively saturated;
  (III) $f(a) < b$ for all partial $\mathcal{N}$-recursive functions; and
  (IV) $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$.

[2]Solovay's paper (as listed in [11]) is entitled *Cuts in models of Peano*. We have not been able to obtain a copy of this paper.

- *Stage 3 Outline*: We use Theorem 2.1 to embed $\mathcal{N}$ onto a proper initial segment J of $\mathcal{M}$. By elementary considerations, this will yield a proper cut I of J with $(\mathcal{M}, \mathcal{A}) \cong (I, \mathcal{A} \restriction I)$.

We now proceed to flesh out the above outlines for the second and third stages.

Stage 2 Details

This stage of the proof can be summarized into a theorem in its own right (Theorem 3.2).[3] Before stating the theorem, we need the following definitions.

3.1. Definition. Suppose $\mathcal{M} \subsetneq_{\text{end}} \mathcal{N}$, where $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $\mathcal{N} \models \mathrm{I}\Delta_0$.

(a) $\in_{\mathsf{Ack}}$ is Ackermann's membership relation [8, 1.31, p.38] based on binary expansions[4].

(b) Suppose $c \in N$ and 2^c exists in $\mathcal{N}$. $c_{E,\mathcal{N}}$ is the '$\in_{\mathsf{Ack}}$-extension' of c in $\mathcal{N}$, i.e.,

$$c_{E,\mathcal{N}} := \{i \in N : \mathcal{N} \models i \in_{\mathsf{Ack}} c\}.$$

Note that since exp holds in $\mathcal{M}$, and $\mathcal{N}$ satisfies $\mathrm{I}\Delta_0$, we can fix some element $k \in N \backslash M$ and apply Δ_0-overspill to the Δ_0-formula

$$\delta(x) := \exists y < k \ (2^x = y)$$

to be assured of the existence of an element $c \in N \backslash M$ for which 2^c exists in $\mathcal{N}$.

(c) $\mathrm{SSy}_M(\mathcal{N})$ is the M-standard system of $\mathcal{N}$, which is defined as the collection of subsets of M that are of the form $M \cap c_{E,\mathcal{N}}$ for some $c \in N$ such that 2^c exists in $\mathcal{N}$.

(d) Suppose $b \in M$, and let $M_{\leq b} := \{a \in M : \mathcal{M} \models a \leq b\}$. We say that $\mathcal{N}$ is a conservative extension of $\mathcal{M}$ with respect to $\Pi_{1,\leq b}$-sentences iff for all Π_1-formulae $\pi(x_0, \cdots, x_{n-1})$ in the language of arithmetic with the displayed free variables, and for all $a_0, \cdots, a_{n-1}$ in $M_{\leq b}$ we have:

$$\mathcal{M} \models \pi(a_0, \cdots, a_{n-1}) \text{ iff } \mathcal{N} \models \pi(a_0, \cdots, a_{n-1}).$$

[3]We are grateful to Keita Yokoyama for pointing out that Theorem 3.2 is of sufficient interest to be explicitly stated.

[4]Recall: $m \in_{\mathsf{Ack}} n$ is defined as "the m-th digit of the binary expansion of n is 1". It is well known that, provably in $\mathrm{I}\Delta_0$, if 2^n exists then (a) n can be written (uniquely) as a sum of powers of 2 (and hence n has a well-defined binary expansion), and (b) $m \in_{\mathsf{Ack}} n$ iff $2 \nmid \lfloor m2^{-n} \rfloor$.

The following theorem supports the existence of the model $\mathcal{N}$ as in the outline of Stage 2. Note that condition (III) of the outline of Stage 2 is implied by the last clause of Theorem 3.2 since if δ is any Δ_0-formula of standard length, and $\overline{a}$ and $\overline{b}$ are constants representing a and b, then the sentence

$$\exists y\ \delta(\overline{a}, y) \to \exists y < \overline{b}\ \delta(\overline{a}, y)$$

is a $\Pi_{1,\leq b}$-sentence .

3.2. Theorem. *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* WKL_0 *and let* $b \in M$*. Then* $\mathcal{M}$ *has a recursively saturated proper end extension* $\mathcal{N}$ *satisfying* $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ *such that* $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$*, and* $\mathcal{N}$ *is a conservative extension of* $\mathcal{M}$ *with respect to* $\Pi_{1,\leq b}$*-sentences.*

Proof: Let $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ be the set of 'true' Π_1-sentences with (parameters in M), as computed in $\mathcal{M}$.[5] Fix some nonstandard $n^* \in M$ with $n^* >> b$, e.g., $n^* = \mathrm{supexp}(b)$ is more than sufficient[6]. Since $\mathcal{M}$ satisfies $\mathrm{I}\Sigma_1$ and $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ has a Π_1-definition within $\mathcal{M}$, there is some element $c \in M$ that codes the fragment of $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ consisting of elements of $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ that are below n^*, i.e.,

$$c_{E,\mathcal{M}} = \{m \in M : m \in \mathbf{True}^{\mathcal{M}}_{\Pi_1} \text{ and } m < n^*\},$$

We observe that $c_{E,\mathcal{M}}$ contains all $\Pi_{1,\leq b}$-sentences that hold in $\mathcal{M}$. Within $\mathcal{M}$, we define the 'theory' T_0 by:

$$T_0 := \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \{m : m \in \mathbf{True}_{\Pi_1} \text{ and } m < n^*\}.$$

At this point we recall a result of Clote-Hájek-Paris [2] that asserts:

(#) $\mathrm{I}\Sigma_1 \vdash \mathsf{Con}(\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathbf{True}_{\Pi_1})$.

In light of (#), we have:

$(*)$ $\mathcal{M} \models \mathsf{Con}(T_0)$.

It is clear that T_0 has a Δ_1-definition in $\mathcal{M}$ (note: here we are not claiming anything about the *quantifier complexity of each member of* T_0). Hence by Δ^0_1-comprehension available in WKL_0 we also have:

$(**)$ $T_0 \in \mathcal{A}$

We wish to build a certain chain $\langle \mathcal{N}_n : n \in \omega \rangle$ of structures such that the elementary diagram of each $(\mathcal{N}_n, a)_{a \in N_n}$ can be computed by $(\mathcal{M}, \mathcal{A})$ as some $E_n \in \mathcal{A}$. Note that E_n would be replete with sentences of nonstandard length. Enumerate $\mathcal{A}$ as $\langle A_n : n \in \omega \rangle$. Our official requirements for the sequence $\langle \mathcal{N}_n : n \in \omega \rangle$ is that for each $n \in \omega$ the following three conditions are satisfied:

[5]It is well-known that $\mathrm{I}\Sigma_1$ has enough power to meaningfully define $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ (see, e.g., [8, Theorem 1.75, p.59]). Note that $\mathbf{True}^{\mathcal{M}}_{\Pi_1}$ need not be in $\mathcal{A}$.

[6]Here $\mathrm{supexp}(b)$ is an exponential stack of length $b + 1$, where the top entry is b, and the rest of the entries consist of 2's.

(1) $T_0 \subseteq E_n \in \mathcal{A}$ (in particular: $\mathcal{N}_n$ is recursively saturated).
(2) $\mathcal{M} \subsetneq_{\text{end}} \mathcal{N}_n \prec \mathcal{N}_{n+1}$.
(3) $A_n \in \mathrm{SSy}_M(\mathcal{N}_{n+1})$.

We now sketch the recursive construction of the desired chain of models. To begin with, we invoke $(*)$, $(**)$, and the completeness theorem for first order logic (that is available in WKL_0, see [13, Theorem IV.3.3]) to get hold of $\mathcal{N}_0$ and E_0 satisfying condition (1). To define $\mathcal{N}_{n+1}$, we assume that we have $\mathcal{N}_n$ satisfying (1). Next, consider the following 'theory' T_{n+1} defined within $(\mathcal{M}, \mathcal{A})$:

$$T_{n+1} := E_n + \left\{\bar{t} \in_{\mathsf{Ack}} d : t \in A_n\right\} + \left\{\bar{t} \notin_{\mathsf{Ack}} d : t \notin A_n\right\},$$

where d is a new constant symbol, and $\bar{t}$ is the numeral representing t in the ambient model $\mathcal{M}$. Note that T_{n+1} belongs to $\mathcal{A}$ since $\mathcal{A}$ is a Turing ideal and T_{n+1} is Turing reducible to the join of E_n and A_n. It is easy to see that T_{n+1} is consistent in the sense of $(\mathcal{M}, \mathcal{A})$ since $(\mathcal{M}, \mathcal{A})$ can verify that T_{n+1} is finitely interpretable in $\mathcal{N}_n$. Using the completeness theorem within $(\mathcal{M}, \mathcal{A})$ this allows us to get hold of the desired $\mathcal{N}_{n+1}$ and E_{n+1} satisfying (1), (2), and (3). The recursive saturation of $\mathcal{N}_{n+1}$ follows immediately from (1), using a well known over-spill argument. Let

$$\mathcal{N} := \bigcup_{n\in\omega} \mathcal{N}_n.$$

It is evident that $\mathcal{N}$ satisfies the properties listed in Theorem 3.2. □

3.3. Remark. The result of Clote-Hájek-Paris that was invoked in the above proof was further extended by Beklemishev, who showed the consistency of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathbf{True}_{\Pi_2}$ within $\mathrm{I}\Sigma_1$ ($n = 1$ of [1, Theorem 5.1]). Beklemishev's result shows that condition $(*)$ of Stage (2) can be strengthened to assert that $\mathcal{M} \models \mathsf{Con}(T_0 + \mathsf{PRA})$, where PRA is the scheme asserting the totality of all primitive recursive functions (recall that the provable recursive functions of $\mathrm{I}\Sigma_1$ are precisely the primitive recursive functions). This has the pleasant consequence that the model $\mathcal{N}$ constructed in Stage 2 can be required to satisfy PRA, and *a fortiori*: $\mathtt{exp}$.

Stage 3 Details

Thanks to properties (I), (II), and (III) of the outline of Stage 2, we can invoke Theorem 2.1 to get hold of a self-embedding ϕ of $\mathcal{N}$ onto a cut J with $a \in J < b$. The image I of M under ϕ is an initial segment of $\mathcal{M}$ and

$$I < J < M.$$

Let $\mathcal{B} := \{\widehat{\phi}(A) : A \in \mathcal{A}\}$. It is clear that ϕ induces an isomorphism

$$\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{B}),$$

Coupled with the fact that $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$, this implies $\mathcal{A} \upharpoonright I = \mathcal{B}$ by elementary considerations. This concludes the proof of (the first clause) of Tanaka's Theorem. □

3.4. Remark. The 'moreover' clause of Tanaka's Theorem can be established by the following variations of the above proof: in Stage 1, given a prescribed $a \in M$, let $a^* = \mathrm{supexp}(a)$ and use strong Σ_1-collection to find $b \in M$ such that:

(1) For every Δ_0-formula $\delta(x, y)$, and every $i \leq a^*$,

$$\mathcal{M} \models \exists y\ \delta(i, y) \Longrightarrow \mathcal{M} \models \exists y < b\ \delta(i, y).$$

Let I be the smallest cut of $\mathcal{M}$ that contains a and is closed under exponentiation, i.e., $I := \{m \in M : \exists n < \omega \text{ such that } m < a_n\}$, where $\langle a_n : n < \omega \rangle$ is given by $a_0 := a$, and $a_{n+1} = 2^{a_n}$. In light of (1), we have:

(2) For every Δ_0-formula $\delta(x, y)$, and every $i \in I$,

$$\mathcal{M} \models \exists y\ \delta(i, y) \Longrightarrow \mathcal{M} \models \exists y < b\ \delta(i, y).$$

Then in Stage 3, we use (2) in conjunction with the following refinement of Theorem 2.1.

3.5. Theorem [5]. *Suppose $\mathcal{N}$ is a countable recursively saturated model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$, I is a cut of $\mathcal{N}$ that is closed under exponentiation, and for some b in $\mathcal{N}$, the following holds for all Δ_0-formulas $\delta(x, y)$:*

$$\forall i \in I\ \ (\mathcal{N} \models \exists y\ \delta(i, y) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(i, y)).$$

There is an isomorphism $\phi : \mathcal{N} \to J$, where J is an initial segment of $\mathcal{N}$ with $J < b$, and $\phi(m) = m$ for all $m \in I$.

The proof of Theorem 3.5 is obtained by dovetailing the proof of Theorem 2.1 with an old argument of Hájek and Pudlák in [7, Appendix].

As pointed out by Keita Yokoyama, the results of this paper, coupled with an observation of Tin Lok Wong, yield the following characterization of models of WKL_0 among countable models of RCA_0.

3.6. Theorem. *Let $(\mathcal{M}, \mathcal{A})$ be a countable model of RCA_0. The following are equivalent:*

(1) *$(\mathcal{M}, \mathcal{A})$ is a model of WKL_0.*

(2) *For every $b \in M$ there exists a recursively saturated proper end extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathsf{PRA}$, $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$, and $\mathcal{N}$ is a conservative extension of $\mathcal{M}$ with respect to $\Pi_{1,\leq b}-$sentences.*

(3) *For every $b \in M$ there is a proper initial segment I of $\mathcal{M}$ such that $(\mathcal{M}, \mathcal{A})$ is isomorphic to $(I, \mathcal{A} \upharpoonright I)$ via an isomorphism that pointwise fixes $M_{\leq b}$.*

Proof. (1) ⇒ (2) is justified by Theorem 3.2 and Remark 3.3, and (2) ⇒ (3) follows from Theorem 3.5. As pointed out by Tin-Lok Wong, (3) ⇒ (1)

follows from the same line of argument as in Ressayre's characterization of models of $I\Sigma_1$ among countable models of $I\Delta_0$ that is mentioned in footnote 1. □

Acknowledgments. The new proof of Tanaka's theorem was discovered by the author in the midst of inspiring e-mail discussions with Volodya Shavrukov on a range of topics concerning self-embeddings of models of arithmetic. More recently, I have also benefitted from conversations with Keita Yokoyama. I am grateful to both gentlemen for offering insightful suggestions for improving the paper.

References

[1] L.D. Beklemishev, *A proof-theoretic analysis of collection*, **Arch. Math. Log**. vol. 37 (1998) pp. 275-296.
[2] P. Clote, P. Hájek, and J. Paris, *On some formalized conservation results in arithmetic*, **Arch. Math. Logic** vol. 30 (1990), pp. 201-218.
[3] C. Cornaros, *Versions of Friedman's theorem for fragments of* PA, manuscript available at:
http://www.softlab.ntua.gr/~nickie/tmp/pls5/cornaros.pdf.
[4] C. Dimitracopoulos and J. Paris, *A note on a theorem of H. Friedman.* **Z. Math. Logik Grundlag**. **Math.** vol. 34 (1988), pp. 13-17.
[5] A. Enayat, *Self-embeddings of models of arithmetic, redux* (in preparation).
[6] H. Friedman, *Countable models of set theories*, **Lecture Notes in Math**., vol. 337, Springer, Berlin, 1973, pp. 539-573.
[7] P. Hájek and P. Pudlák, *Two orderings of the class of all countable models of Peano arithmetic*, **Model theory of algebra and arithmetic,** Lecture Notes in Math, vol. 834, 1980, pp. 174-185.
[8] ———, **Metamathematics of First Order Arithmetic**, Springer, Heidelberg, 1993.
[9] R. Kaye, **Models of Peano Arithmetic**, Oxford Logic Guides, Oxford University Press, Oxford, 1991.
[10] H. Lessan, **Models of Arithmetic**, Doctoral Dissertation, University of Manchester, 1978.
[11] J. B. Paris, *Some conservation results for fragments of arithmetic*, in **Lecture Notes in Math**., vol. 890, Springer, Berlin-New York, 1981, pp. 251-262.
[12] J.-P. Ressayre, *Nonstandard universes with strong embeddings, and their finite approximations.* **Logic and combinatorics** in Contemp. Math**.,** vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 333-358.
[13] S. Simpson, **Subsystems of second order arithmetic**, Springer-Verlag, 1999.
[14] K. Tanaka, *The self-embedding theorem of* WKL_0 *and a non-standard method*, **Ann. Pure Appl. Logic** vol. 84 (1997), pp. 41-49.
[15] K. Tanaka and T. Yamazaki, *A Non-Standard Construction of Haar Measure and Weak König's Lemma,* **J. Symbolic Logic**, vol. 65 (2000), pp. 173-186.
[16] K. Yokoyama, *Various versions of self-embedding theorem*, manuscript (May 2012).