MULTIPLIERS OF MULTIDIMENSIONAL FOURIER ALGEBRAS

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Abstract. Let G be a locally compact σ-compact group. Motivated by an earlier notion for discrete groups due to Effros and Ruan, we introduce the multidimensional Fourier algebra $A^n(G)$ of G. We characterise the completely bounded multidimensional multipliers associated with $A^n(G)$ in several equivalent ways. In particular, we establish a completely isometric embedding of the space of all n-dimensional completely bounded multipliers into the space of all Schur multipliers on G^{n+1} with respect to the (left) Haar measure. We show that in the case G is amenable the space of completely bounded multidimensional multipliers coincides with the multidimensional Fourier-Stieltjes algebra of G introduced by Ylinen. We extend some well-known results for abelian groups to the multidimensional setting.

1. Introduction

A classical result in Harmonic Analysis asserts that a bounded function defined on a locally compact abelian group G is a multiplier of the Fourier algebra $A(G)$ of G precisely when it is the Fourier transform of a regular Borel measure on the character group $\hat{G}$ of G. After the seminal work of P. Eymard [10], Harmonic Analysis on general locally compact groups has been closely related to the theory of C*- and von Neumann algebras. More recent work of E. Effros, M. Neufang, Zh.-J. Ruan, V. Runde, N. Spronk and others shows that Operator Space Theory plays a significant role in the subject. The operator space structure of $A(G)$ has thus become an indispensable tool in non-commutative Harmonic Analysis. J. de Cannière and U. Haagerup [4] defined the set $M^{cb}A(G)$ of completely bounded multipliers of $A(G)$, and M. Bozejko and G. Fendler [3] provided a characterisation of $M^{cb}A(G)$ which, combined with a classical result of A. Grothendieck [13] and a result of V. Peller [17] shows that $M^{cb}A(G)$ can be isometrically identified with the space of all Schur multipliers of Toeplitz type. An alternative proof of this result was given by P. Jolissaint [14]. N. Spronk [21] showed that this

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identification is in fact a complete isometry. We refer the reader to Sections 5 and 6 of G. Pisier's monograph [18] for an account of Schur multipliers.

Building on an earlier work on bimeasures on locally compact groups [11], [12], K. Ylinen [22] defined a multivariable version $B^n(G)$ of the Fourier-Stieltjes algebra of a locally compact group. A multivariable version of the Fourier algebra of a discrete group was introduced by E. Effros and Zh.-J. Ruan in [7], and its completely bounded multipliers were characterised in terms of a multilinear matrix version of classical Schur multipliers, introduced in the same paper.

In [15], multidimensional Schur multipliers associated with measure spaces were introduced and identified with a natural extended Haagerup tensor product [9] up to an isometry. In the present paper, we show that this identification is a complete isometry. We define the n-dimensional Fourier algebra $A^n(G)$ of an arbitrary locally compact group and show that it is a closed ideal of $B^n(G)$. We characterise the set $M_n^{cb}A(G)$ of completely bounded multipliers associated with $A^n(G)$ in several equivalent ways (Proposition 5.4, Theorem 5.5, Theorem 5.7). In particular, we show that there exists a completely isometric inclusion of $M_n^{cb}A(G)$ into the space of all $n+1$-dimensional Schur multipliers on G with respect to the (left) Haar measure. Its image is a space of multidimensional Schur multipliers of Toeplitz type. Our results imply that if G is amenable then $B^n(G)$ can be completely isometrically identified with $M_n^{cb}A(G)$. In the case G is abelian, we show that $B^n(G)$ can be identified with more general classes of multipliers on G arising from partitions of the variables (Theorem 6.4). In particular, every multiplier of $A^n(G)$ is in this case automatically completely bounded. We obtain a multidimensional version of the classical result that if $\varphi \in \ell^\infty(\mathbb{Z})$ then the function $\tilde{\varphi} \in \ell^\infty(\mathbb{Z} \times \mathbb{Z})$ given by $\tilde{\varphi}(x,y) = \varphi(x-y)$ is a Schur multiplier if and only if φ is the Fourier transform of a regular Borel measure on the unit circle.

2. Preliminaries

We begin by recalling some basic notions and results from Eymard's work [10]. If H and K are Hilbert spaces we let $\mathcal{B}(H,K)$ be the space of all bounded linear operators from H into K. We write $\mathcal{B}(H) = \mathcal{B}(H,H)$. Throughout the paper, G will denote a locally compact σ-compact group with a left Haar measure m and a neutral element e. As usual, $L^p(G)$, $p = 1, 2$, will denote the space of all complex valued Borel functions f on G such that $|f|^p$ is integrable with respect to m.

The space $L^1(G)$ is an involutive Banach algebra; its enveloping C*-algebra is the *group C*-algebra* $C^*(G)$ of G. We denote by $W^*(G)$ the enveloping von Neumann algebra of $C^*(G)$ and let $\omega : G \to W^*(G)$ be the canonical homomorphism of G into $W^*(G)$. Let λ be the left regular representation of $L^1(G)$ on the Hilbert space $L^2(G)$; the closure of its image in the operator norm is the *reduced C*-algebra* $C_r^*(G)$ of G, and its closure in the weak operator topology is the *group von Neumann algebra* $\mathrm{VN}(G)$ of G. We use the symbol λ to also denote the left regular representation of G on $L^2(G)$.

Let $B(G) = C^*(G)^*$ be the *Fourier-Stieltjes algebra* of G; if $f \in B(G)$ then f can be identified with a function (denoted in the same way and) given by $f(x) = \langle f, \omega(x)\rangle$. Any such f has the form $f(x) = (\pi(x)\xi, \eta)$ for some unitary representation $\pi : G \to \mathcal{B}(H)$ and vectors $\xi, \eta \in H$, and the space $B(G)$ is a Banach algebra with respect to the pointwise product. By $A(G)$ we denote as usual the *Fourier algebra* of G, that is, the ideal of $B(G)$ of all functions f of the form $f(x) = (\lambda_x\xi, \eta)$ where $\xi, \eta \in L^2(G)$. Then $A(G)$ can be canonically identified with the predual of $\mathrm{VN}(G)$: if $f(x) = (\lambda_x\xi, \eta)$, $x \in G$, then $\langle f, T\rangle = (T\xi, \eta)$, $T \in \mathrm{VN}(G)$.

We next recall some notions and facts from Operator Space Theory. We refer the reader to [1], [8], [16] and [19] for further details. An *operator space* is a closed subspace $\mathcal{E}$ of $\mathcal{B}(H, K)$ for some Hilbert spaces H and K. If $n, m \in \mathbb{N}$, we will denote by $M_{n,m}(\mathcal{E})$ the space of all n by m matrices with entries in $\mathcal{E}$ and let $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$. Note that $M_{n,m}(\mathcal{E})$ can be identified in a natural way with a subspace of $\mathcal{B}(H^m, K^n)$ and hence carries a natural operator norm. If $n = \infty$ or $m = \infty$, we will denote by $M_{n,m}(\mathcal{E})$ the space of all (singly or doubly infinite) matrices with entries in $\mathcal{E}$ which represent a bounded linear operator between the corresponding amplifications of the Hilbert spaces and set $M_\infty(\mathcal{E}) = M_{\infty,\infty}(\mathcal{E})$. We also write $M_{n,m} = M_{n,m}(\mathbb{C})$ and $M_\infty = M_{\infty,\infty}(\mathbb{C})$. If $\mathcal{E}$ and $\mathcal{F}$ are operator spaces, a linear map $\Phi : \mathcal{E} \to \mathcal{F}$ is called *completely bounded* if the map $\Phi^{(k)} : M_k(\mathcal{E}) \to M_k(\mathcal{F})$, given by $\Phi^{(k)}((a_{ij})) = (\Phi(a_{ij}))$, is bounded for each $k \in \mathbb{N}$ and $\|\Phi\|_{cb} \stackrel{def}{=} \sup_k \|\Phi^{(k)}\| < \infty$. The map Φ is called a *complete isometry* if $\Phi^{(k)}$ is an isometry for each $k \in \mathbb{N}$, and a *complete contraction* if $\|\Phi\|_{cb} \leq 1$.

If $\mathcal{E}$ (resp. $\mathcal{F}$) is a linear space and $\|\cdot\|_k$ is a norm on $M_k(\mathcal{E})$ (resp. $M_k(\mathcal{F})$), $k \in \mathbb{N}$, then one may speak of completely bounded, completely contractive and completely isometric mappings from $\mathcal{E}$ into $\mathcal{F}$ as described above. Ruan's celebrated abstract characterisation of operator spaces identifies a set of axioms on the family $(\|\cdot\|_k)_{k=1}^\infty$ of norms in order that $\mathcal{E}$ be completely isometric to an operator space; see [8] for a description of these axioms and applications. An *operator space structure* on a linear space $\mathcal{E}$ is a family $(\|\cdot\|_k)_{k=1}^{\infty}$, where $\|\cdot\|_k$ is a norm on $M_k(\mathcal{E})$, with respect to which $\mathcal{E}$ is completely isometric to an operator space.

Let $\mathcal{E}, \mathcal{E}_1, \dots, \mathcal{E}_n$ be operator spaces, $\Phi : \mathcal{E}_1 \times \dots \times \mathcal{E}_n \to \mathcal{E}$ be a multilinear map and

$$\Phi^{(k)} : M_k(\mathcal{E}_1) \times M_k(\mathcal{E}_2) \times \dots \times M_k(\mathcal{E}_n) \to M_k(\mathcal{E})$$

be the multilinear map given by

$$\Phi^{(k)}(a^1, \dots, a^n)_{p,q} = \sum_{p_2,\dots,p_n} \Phi(a^1_{p,p_2}, a^2_{p_2,p_3}, \dots, a^n_{p_n,q}), \tag{1}$$

where $a^i = (a^i_{p,q}) \in M_k(\mathcal{E}_i)$, $1 \leq p, q \leq k$. The map Φ is called completely bounded if there exists $C > 0$ such that for all $k \in \mathbb{N}$ and all elements $a^i \in M_k(\mathcal{E}_i)$, $i = 1, \dots, n$, we have

$$\|\Phi^{(k)}(a^1, \dots, a^n)\| \leq C\|a^1\| \dots \|a^n\|.$$

If $\mathcal{E}$ and $\mathcal{E}_i$, $i = 1, \dots, n$, are dual operator spaces we say that Φ is *normal* if it is weak* continuous in each variable. We denote by $CB^{\sigma}(\mathcal{E}_1 \times \dots \times \mathcal{E}_n, \mathcal{E})$ the set of all normal completely bounded multilinear maps from $\mathcal{E}_1 \times \dots \times \mathcal{E}_n$ into $\mathcal{E}$; this space can be equipped with an operator space structure in a canonical way (see [9]).

E. Christensen and A. Sinclair [6] gave a characterisation of completely bounded (resp. normal completely bounded) multilinear maps defined on the direct product of finitely many C*-algebras (resp. von Neumann algebras). We will need the following generalisation of Corollaries 5.7 and 5.9 of [6] whose proof is a straightforward generalisation of the proof of Corollary 5.9 of [6]. If $\mathcal{A}$ is a set we let $\mathcal{A}^n = \underbrace{\mathcal{A} \times \dots \times \mathcal{A}}_{n}$. If $\mathcal{M}$ is a von Neumann algebra and $\mathcal{R}_j \subseteq \mathcal{M}$, $j = 1, \dots, n-1$, are von Neumann subalgebras, we say that a mapping $\Phi : \mathcal{M}^n \to B(H)$ is $(\mathcal{R}_1, \dots, \mathcal{R}_{n-1})$-modular if

$$\Phi(a_1 r_1, a_2 r_2, \dots, a_n) = \Phi(a_1, r_1 a_2, \dots, r_{n-1} a_n),$$

for all $a_1, \dots, a_n \in \mathcal{M}$, $r_j \in \mathcal{R}_j$, $j = 1, \dots, n-1$.

Theorem 2.1. *Let $\mathcal{M} \subseteq \mathcal{B}(K)$ be a von Neumann algebra, $\mathcal{R}_j \subseteq \mathcal{M}$ be a von Neumann subalgebra, $j = 1, \dots, n-1$, H be a Hilbert space and $\Phi : \mathcal{M}^n \to \mathcal{B}(H)$ be a multilinear map. The following are equivalent:*

(i) Φ is completely bounded, normal and $(\mathcal{R}_1, \dots, \mathcal{R}_{n-1})$-modular;

(ii) there exists an index set J and operators $V_j \in M_J(\mathcal{R}_j')$, $j = 1, \dots, n-1$, $V_0 \in \mathcal{B}(K^J, H)$ and $V_n \in M_{1,J}(H, K^J)$ such that for all

$a_1, \dots, a_n \in \mathcal{M}$, *we have*

$$\Phi(a_1, \dots, a_n) = V_0(a_1 \otimes 1_J)V_1 \dots V_{n-1}(a_n \otimes 1_J)V_n.$$

Moreover, if (i) holds then $\|\Phi\|_{cb}$ *equals the infimum of* $\|V_0\| \dots \|V_n\|$ *over all representations of* Φ *as in (ii) and this infimum is attained.*

Tensor products will play a substantial role in the paper. We denote by $V \odot W$ the algebraic tensor product of the vector spaces V and W. If $\mathcal{E}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{E}_2 \subseteq \mathcal{B}(H_2)$ are operator spaces and $u \in \mathcal{E}_1 \odot \mathcal{E}_2$, the *Haagerup norm* of u is given by

$$\|u\|_h = \inf\left\{ \left\| \sum_{j=1}^k a_j a_j^* \right\|^{\frac{1}{2}} \left\| \sum_{j=1}^k b_j^* b_j \right\|^{\frac{1}{2}} : u = \sum_{j=1}^k a_j \otimes b_j \right\}.$$

The completion $\mathcal{E}_1 \otimes_h \mathcal{E}_2$ of $\mathcal{E}_1 \odot \mathcal{E}_2$ with respect to $\|\cdot\|_h$ is the *Haagerup tensor product* of $\mathcal{E}_1$ and $\mathcal{E}_2$. We refer the reader to [8] for its properties and to [9] for the definition and properties of the extended Haagerup tensor product $\mathcal{E}_1 \otimes_{eh} \mathcal{E}_2$ and the normal Haagerup tensor product $\mathcal{E}_1 \otimes_{\sigma h} \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. We recall the canonical identifications $(\mathcal{E}_1 \otimes_h \mathcal{E}_2)^* = \mathcal{E}_1^* \otimes_{eh} \mathcal{E}_2^*$ and $(\mathcal{E}_1 \otimes_{eh} \mathcal{E}_2)^* = \mathcal{E}_1^* \otimes_{\sigma h} \mathcal{E}_2^*$. If $\delta \in \mathcal{E}_1^*$ then the left slice map $L_\delta : \mathcal{E}_1 \otimes_{eh} \mathcal{E}_2 \to \mathcal{E}_2$ is the unique completely bounded map given on elementary tensors by $L_\delta(a \otimes b) = \delta(a)b$ [9]. Similarly, for $\delta \in \mathcal{E}_2^*$ one defines the right slice map $R_\delta : \mathcal{E}_1 \otimes_{eh} \mathcal{E}_2 \to \mathcal{E}_1$.

If $\mathcal{X}$ is a Banach space we denote by $b_1(\mathcal{X})$ the unit ball of $\mathcal{X}$. Banach space duality is denoted by $\langle \cdot, \cdot \rangle$. We denote by 1_H the identity operator on a Hilbert space H and, for a cardinal J, write $1_J = 1_{\ell^2(J)}$. The identity operator on $\ell^2(\mathbb{N})$ is often denoted simply by 1.

3. The operator space of Schur multipliers

In this section we recall the definition of multidimensional Schur multipliers associated with measure spaces and prove a completely isometric version of the characterisation result, Theorem 3.4, of [15].

Let (X_i, μ_i), $i = 1, \dots, n$, be standard measure spaces and

$$\Gamma(X_1, \dots, X_n) = L^2(X_1 \times X_2) \odot \dots \odot L^2(X_{n-1} \times X_n),$$

where the direct products are equipped with the corresponding product measures. We identify the elements of $\Gamma(X_1, \dots, X_n)$ with functions on

$$X_1 \times X_2 \times X_2 \times \dots \times X_{n-1} \times X_{n-1} \times X_n$$

in the obvious fashion. We equip $\Gamma(X_1, \dots, X_n)$ with the Haagerup tensor norm $\|\cdot\|_{\rm h}$, where the L^2-spaces are given their opposite operator space structure (see [19]) arising from the identification $f \longleftrightarrow T_f$ of $L^2(X \times Y)$ with the class of Hilbert-Schmidt operators from $L^2(X)$ into

$L^2(Y)$ where, for $f \in L^2(X \times Y)$, we let T_f be the (Hilbert-Schmidt) operator given by

$$(T_f\xi)(y) = \int_X f(x,y)\xi(x)dx, \qquad \xi \in L^2(X),\ y \in Y, \tag{2}$$

dx denoting integration with respect to μ. If $f \in L^2(X \times Y)$ we let $\|f\|_{\rm op}$ be equal to the operator norm of T_f.

For each $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ let

$$S_\varphi : \Gamma(X_1, \ldots, X_n) \to L^2(X_1 \times X_n)$$

be the map sending $f_1 \otimes \cdots \otimes f_{n-1} \in \Gamma(X_1, \ldots, X_n)$ to the function which maps (x_1, x_n) to

$$\int \varphi(x_1, \ldots, x_n) f_1(x_1, x_2) f_2(x_2, x_3) \ldots f_{n-1}(x_{n-1}, x_n) dx_2 \ldots dx_{n-1}.$$

It was shown in Theorem 3.1 of [15] that S_φ is a bounded mapping when $\Gamma(X_1, \ldots, X_n)$ is equipped with the projective norm where each of its terms is given the L^2-norm, and that $\|S_\varphi\| = \|\varphi\|_\infty$.

Definition 3.1. *A function $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ is called a Schur multiplier (relative to the measure spaces $(X_1, \mu_1), \ldots (X_n, \mu_n)$) if there exists $C > 0$ such that $\|S_\varphi(u)\|_{\rm op} \leq C\|u\|_{\rm h}$, for all $u \in \Gamma(X_1, \ldots, X_n)$. The smallest constant C with this property is denoted by $\|\varphi\|_{\rm m}$.*

Let $H_i = L^2(X_i)$, $i = 1, \ldots, n$, and $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ be a Schur multiplier. It was shown in Section 3 of [15] that φ induces a normal completely bounded multilinear map

$$\tilde{S}_\varphi : \mathcal{B}(H_{n-1}, H_n) \times \cdots \times \mathcal{B}(H_1, H_2) \to \mathcal{B}(H_1, H_n)$$

such that $\|\tilde{S}_\varphi\|_{cb} = \|\varphi\|_{\rm m}$ and $\tilde{S}_\varphi(T_{f_{n-1}}, \ldots, T_{f_1}) = S_\varphi(f_1 \otimes \cdots \otimes f_{n-1})$, for all $f_i \in L^2(X_i \times X_{i+1})$, $i = 1, \ldots, n$. We denote by $\mathcal{S} = \mathcal{S}(X_1, \ldots, X_n)$ the collection of all Schur multipliers in $L^\infty(X_1 \times \cdots \times X_n)$. It follows that $\mathcal{S}$ can be canonically embedded into $CB^\sigma(\mathcal{B}(H_{n-1}, H_n) \times \cdots \times \mathcal{B}(H_1, H_2), \mathcal{B}(H_1, H_n))$. Thus, $\mathcal{S}$ inherits an operator space structure from the latter space. More precisely, if $\varphi = (\varphi_{p,q}) \in M_k(\mathcal{S})$ we have $\|\varphi\|_{{\rm m},k} \stackrel{def}{=} \|(\tilde{S}_{\varphi_{p,q}})\|_{\rm cb}$, where $\tilde{S}_\varphi = (\tilde{S}_{\varphi_{p,q}})$ is identified with a normal completely bounded multilinear map from $\mathcal{B}(H_{n-1}, H_n) \times \cdots \times \mathcal{B}(H_1, H_2)$ into $M_k(\mathcal{B}(H_1, H_n))$. Note that a matrix $\varphi = (\varphi_{p,q}) \in M_k(\mathcal{S})$ can be viewed as a map $\varphi : X_1 \times \cdots \times X_n \to M_k$ by letting $\varphi(x_1, \ldots, x_n) = (\varphi_{p,q}(x_1, \ldots, x_n)) \in M_k$.

The following result is a matricial version of Theorem 3.4 of [15].

Theorem 3.2. *Let $\varphi = (\varphi_{p,q}) \in M_k(\mathcal{S})$. The following are equivalent:*

(i) $\|\varphi\|_{{\rm m},k} < 1$;

(ii) there exist essentially bounded functions $a_1 : X_1 \to M_{\infty,k}$, $a_n : X_n \to M_{k,\infty}$ and $a_i : X_i \to M_\infty$, $i = 2, \dots, n-1$, such that, for almost all $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$, we have

$$\varphi(x_1, \dots, x_n) = a_n(x_n)a_{n-1}(x_{n-1}) \dots a_1(x_1) \text{ and } \operatorname{esssup}_{x_i \in X_i} \prod_{i=1}^{n} \|a_i(x_i)\| < 1.$$

Proof. (i)⇒(ii) Let $\mathcal{D}_i$ be the multiplication masa of $L^\infty(X_i)$. The proof of Theorem 3.4 of [15] implies that the mapping

$$\tilde{S}_\varphi \stackrel{def}{=} (\tilde{S}_{\varphi_{p,q}}) : \mathcal{B}(H_{n-1}, H_n) \times \dots \times \mathcal{B}(H_1, H_2) \to M_k(\mathcal{B}(H_1, H_n))$$

is normal, completely bounded, and $(\mathcal{D}_n, \dots, \mathcal{D}_1)$-modular in the sense that

$$\tilde{S}_\varphi(A_n T_{n-1} A_{n-1}, \dots, T_1 A_1) =$$
$$(A_n \otimes 1_k)\tilde{S}_\varphi(T_{n-1}, A_{n-1}T_{n-2}, \dots, A_2 T_1)(A_1 \otimes 1_k),$$

whenever $A_i \in \mathcal{D}_i$, $i = 1, \dots, n$. A modification of Corollary 5.9 of [6] shows that there exist operators $V_1 : H_1^k \to H_1^\infty$, $V_i : H_i^\infty \to H_i^\infty$, $i = 2, \dots, n-1$ and $V_n : H_n^\infty \to H_n^k$ such that the entries of V_i belong to $\mathcal{D}_i$, $\Pi_{i=1}^n \|V_i\| < 1$ and

$$\tilde{S}_\varphi(T_{n-1}, \dots, T_1) = V_n(T_{n-1} \otimes I) \dots (T_1 \otimes I)V_1,$$

for all $T_i \in \mathcal{B}(H_i, H_{i+1})$, $i = 1, \dots, n$. If $V_i = (A^i_{s,t})_{s,t}$, where $A^i_{s,t}$ is the multiplication operator corresponding to $a^i_{s,t} \in L^\infty(X_i)$ let $a_i : X_i \to M_\infty$ be the function given by $a_i(x_i) = (a^i_{s,t}(x_i))_{s,t}$, $x_i \in X_i$, $i = 1, \dots, n$. Define $a_1 : X_1 \to M_{\infty,k}$ and $a_n : X_n \to M_{k,\infty}$ similarly. Then $\operatorname{esssup}_{x_i \in X_i} \prod_{i=1}^n \|a_i(x_i)\| = \Pi_{i=1}^n \|V_i\| < 1$.

Let V_n^p (resp. V_1^q) be the pth row (resp. the qth column) of V_n (resp. V_1). Let $a_n^p : X_n \to M_{1,\infty}$ (resp. $a_1^q : X_1 \to M_{\infty,1}$) be the function corresponding to V_n^p (resp. V_1^q). We have that

$$\tilde{S}_{\varphi_{p,q}}(T_{n-1}, \dots, T_1) = V_n^p(T_{n-1} \otimes I)V_{n-1} \dots V_2(T_1 \otimes I)V_1^q,$$

for all $T_i \in \mathcal{B}(H_i, H_{i+1})$, $i = 1, \dots, n-1$. It follows from Theorem 3.4 of [15] that

$$\varphi_{p,q}(x_1, \dots, x_n) = a_n^p(x_n)a_{n-1}(x_{n-1}) \dots a_2(x_2)a_1^q(x_1), \quad \text{a.e. } x_i \in X_i.$$

Since this holds for all $p, q = 1, \dots, k$, we have that

$$\varphi(x_1, \dots, x_n) = a_n(x_n)a_{n-1}(x_{n-1}) \dots a_2(x_2)a_1(x_1)$$

for almost all $x_i \in X_i$, $i = 1, \dots, n$.

(ii)⇒(i) In the notation of (i) we have that

$$\varphi_{p,q}(x_1, \dots, x_n) = a_n^p(x_n)a_{n-1}(x_{n-1}) \dots a_2(x_2)a_1^q(x_1),$$

for almost all $x_i \in X_i$, $i = 1, \dots, n$, which in turn implies that

$$\tilde{S}_{\varphi_{p,q}}(T_{n-1}, \dots, T_1) = V_n^p(T_{n-1} \otimes I)V_{n-1} \dots V_2(T_1 \otimes I)V_1^q,$$

and hence that

$$\tilde{S}_{\varphi}(T_{n-1}, \dots, T_1) = V_n(T_{n-1} \otimes I)V_{n-1} \dots V_2(T_1 \otimes I)V_1,$$

for all $T_i \in \mathcal{B}(H_i, H_{i+1})$, $i = 1, \dots, n-1$. It follows that $\|S_\varphi\| < 1$ and so $\|\varphi\|_{\mathrm{m},k} < 1$. $\diamondsuit$

Remark 3.3. Theorem 3.2 amounts to the statement that the identification of the set of all n-dimensional Schur multipliers on $X_1 \times \dots \times X_n$ with the extended Haagerup tensor product $L^\infty(X_n) \otimes_{eh} \dots \otimes_{eh} L^\infty(X_1)$ discussed in the remark after Theorem 3.4 of [15] is completely isometric.

4. The multidimensional Fourier-Stieltjes algebra

In this section we recall the notion of the Fourier transform of a completely bounded multilinear map on the direct product of finitely many group C*-algebras studied in [22], which will provide the basis for our study of multidimensional multipliers. We discuss a description of the multidimensional Fourier-Stieltjes algebra in terms of tensor products and explain its relation to the one dimensional case as well as to the notion of a bimeasure studied in [11].

Let $n \in \mathbb{N}$. An n-**measure** on G is a completely bounded multilinear map $\Phi : C^*(G)^n \to \mathbb{C}$. We note that the term "bimeasure" was used in [11] to designate a bounded bilinear form on $C_0(G) \times C_0(H)$, where G and H are locally compact groups. We will show below that in the case $H = G$ is abelian, the notion of a bimeasure agrees with that of a 2-measure.

We let $M^n(G)$ denote the space of all n-measures on G; by the universal property of the Haagerup tensor product, we have that

$$M^n(G) \equiv \Big(\underbrace{C^*(G) \otimes_h \cdots \otimes_h C^*(G)}_{n}\Big)^*.$$

We equip $M^n(G)$ with the standard operator space structure of a dual operator space arising from the above identification. Suppose that $\Phi \in M^n(G)$. It is standard (see p.156 of [22]) to extend Φ to a normal completely bounded map $\tilde{\Phi} : \underbrace{W^*(G) \otimes_{\sigma h} \cdots \otimes_{\sigma h} W^*(G)}_{n} \to \mathbb{C}$.

Let

$B^n(G) = \{f \in L^\infty(G^n) :$ there exists $\Phi \in M^n(G)$ such that

$$f(x_1,\dots,x_n)=\tilde{\Phi}(\omega(x_1),\dots,\omega(x_n)),\ x_1,\dots,x_n\in G\}. \tag{3}$$

Since $\{\omega(x):x\in G\}$ generates $W^*(G)$ as a von Neumann algebra, we have that the element $\Phi\in M^n(G)$ associated with $f\in B^n(G)$ in (3) is unique. We call f the Fourier transform of Φ and write $f=\hat{\Phi}$. Thus, $B^n(G)$ is in one-to-one correspondence with $M^n(G)$; we equip it with the operator space structure arising from this correspondence. Thus, if $(f_{p,q})\in M_k(B^n(G))$ and $\Phi_{p,q}\in M^n(G)$ is such that $\hat{\Phi}_{p,q}=f_{p,q}$, we have that $\|(f_{p,q})\|_{M_k(B^n(G))}=\|(\Phi_{p,q})\|_{M_k(M^n(G))}$. Since the map $x\to\omega(x)$ is weak* continuous, the space $B^n(G)$ consists of continuous functions. By Corollary 5.4 of [22], $B^n(G)$ is closed under the pointwise product. By [2],

$$B^n(G)\equiv\underbrace{B(G)\otimes_{eh}\cdots\otimes_{eh}B(G)}_{n} \tag{4}$$

up to a complete isometry. We note that if $f\in B^n(G)$ and $a_i\in L^1(G)$, $i=1,\dots,n$, then

$$\langle a_1\otimes\cdots\otimes a_n,f\rangle=\int_{G^n}f(x_1,\dots,x_n)a_1(x_1)\dots a_n(x_n)dm(x_1)\dots dm(x_n). \tag{5}$$

Indeed, (5) is obviously true if f is an elementary tensor, and by linearity, if f is in the algebraic tensor product of n copies of $B(G)$. If $f\in B^n(G)$ then there exists a bounded net $\{f_\nu\}_\nu$ in the algebraic tensor product of n copies of $B(G)$ which tends to f in the topology determined by the duality between $B^n(G)$ and $\underbrace{W^*(G)\odot\cdots\odot W^*(G)}_{n}$ [9]. But then

$$\begin{aligned} f_\nu(x_1,\dots,x_n) &= \langle f_\nu,\omega(x_1)\otimes\cdots\otimes\omega(x_n)\rangle\\ &\to \langle f,\omega(x_1)\otimes\cdots\otimes\omega(x_n)\rangle=f(x_1,\dots,x_n)\end{aligned}$$

for all $x_1,\dots,x_n\in G$ and (5) follows from the Lebesgue Dominated Convergence Theorem.

The following fact proved in [22] will be of importance to us.

Theorem 4.1. [22] *A function f belongs to $B^n(G)$ if and only if there exist a Hilbert space H, vectors $\xi,\eta\in H$ and continuous unitary representations π_i of G on H, $i=1,\dots,n$, such that*

$$f(x_n,\dots,x_1)=(\pi_n(x_n)\dots\pi_1(x_1)\xi,\eta),\quad x_1,\dots,x_n\in G.$$

Moreover, the norm of f equals the infimum of the products $\|\xi\|\|\eta\|$ over all representations of f of the above form.

Theorem 4.1 has the following consequence.

Proposition 4.2. *The multiplication in $B^n(G)$ is completely contractive.*

Proof. Let $(f_{p,q}), (g_{p,q}) \in M_k(B^n(G))$ and $\Phi_{p,q}$ (resp. $\Psi_{p,q}$) be the n-measure such that $\hat{\Phi}_{p,q} = f_{p,q}$ (resp. $\hat{\Psi}_{p,q} = g_{p,q}$). Let $\Phi = (\Phi_{p,q})$ and $\Psi = (\Psi_{p,q})$; then Φ and Ψ can be viewed as completely bounded mappings from $C^*(G)^n$ into M_k. Moreover, $\|(f_{p,q})\|_{M_k(B^n(G))} = \|\Phi\|_{cb}$ and $\|(g_{p,q})\|_{M_k(B^n(G))} = \|\Psi\|_{cb}$.

Let $h_{p,q} = \sum_{r=1}^k f_{p,r}g_{r,q}$ and $\Omega_{p,q} : C^*(G)^n \to M_k$ be the map given by

$$\Omega_{p,q}(a_1, \dots, a_n) = \sum_{r=1}^{k} \Phi_{p,r}(a_1, \dots, a_n)\Psi_{r,q}(a_1, \dots, a_n)$$

(the product on the right hand side being that of M_k). Then $\tilde{\Omega}_{p,q}$ is given in the same way as $\Omega_{p,q}$, with $\Phi_{p,r}$ and $\Psi_{r,q}$ replaced by $\tilde{\Phi}_{p,r}$ and $\tilde{\Psi}_{r,q}$, respectively. Moreover, $\hat{\Omega}_{p,q} = h_{p,q}$. It is clear that if $\Omega = (\Omega_{p,q})$ then $\|\Omega\|_{cb} \leq \|\Phi\|_{cb}\|\Psi\|_{cb}$. The claim follows. $\diamondsuit$

We note that Theorem 4.1 implies that $B^1(G)$ coincides with the Fourier-Stieltjes algebra $B(G)$ of the group G introduced by Eymard [10].

Suppose that G is abelian and $n = 2$. In this case $M^2(G)$ coincides with the set of all bimeasures on the character group $\hat{G}$ of G studied in [12], while $B^2(G)$ coincides with the set of their Fourier transforms. Indeed, let $\Phi \in M^2(G)$. Since G is abelian, $C^*(G)$ is canonically *-isomorphic to $C_0(\hat{G})$. Thus, Φ can be considered as a bounded bilinear form on $C_0(\hat{G}) \times C_0(\hat{G})$ (in other words, a *bimeasure* on $\hat{G}$ in the sense of [12]). On the other hand, for any locally compact Hausdorff space X there exists a canonical injection $\iota : \mathcal{L}^\infty(X) \to C_0(X)^{**}$ (where $\mathcal{L}^\infty(X)$ is the algebra of all bounded Borel functions on X) given by $\iota(f)(\mu) = \int_X f d\mu$, $\mu \in C_0(X)^*$. Let $\Phi_1 : \mathcal{L}^\infty(\hat{G}) \times \mathcal{L}^\infty(\hat{G}) \to \mathbb{C}$ be the extension of Φ described in Corollary 1.3 of [12]. If $x \in G$ let $\check{x}$ be the character of $\hat{G}$ corresponding to x^{-1}. It is straightforward to check that

$$\iota(\check{x}) = \omega(x). \tag{6}$$

We next observe that

$$\tilde{\Phi}(\iota(f), \iota(g)) = \Phi_1(f, g), \quad f, g \in \mathcal{L}^\infty(\hat{G}). \tag{7}$$

To this end, let μ_1 and μ_2 be probability measures associated with Φ through Grothendieck's inequality and let $\{f_\alpha\} \subseteq C_0(\hat{G})$ and $\{g_\alpha\} \subseteq C_0(\hat{G})$ be bounded nets such that $f_\alpha \to \iota(f)$ and $g_\alpha \to \iota(g)$ in the weak* topology of $W^*(G)$. Then $f_\alpha \to f$ in $L^2(\hat{G}, \mu_1)$ and $g_\alpha \to g$ in $L^2(\hat{G}, \mu_2)$. By the definition of $\Phi_1(f, g)$ (see [12]), we have that it is the limit of the net $\{\Phi(f_\alpha, g_\alpha)\}_\alpha$. Identity (7) now follows by approximation.

Now note that (6) and (7) imply

$$\Phi_1(\check{x}, \check{y}) = \tilde{\Phi}(\omega(x), \omega(y)), \qquad x, y \in G.$$

It follows from Definition 1.10 of [12] that $B^2(G)$ coincides with the set of all Fourier transforms of bimeasures on $\hat{G}$.

5. Multipliers of $A^n(G)$: non-abelian groups

In this section, we introduce the multidimensional Fourier algebra $A^n(G)$ of a locally compact group G. For each partition $\mathcal{P}$ of the set $\{n, \dots, 1\}$ into k subsets, we define a completely isometric embedding of $A^k(G)$ into $A^n(G)$. Using these embeddings, we define the (completely bounded) multipliers of G relative to $\mathcal{P}$. We characterise the completely bounded multipliers corresponding to the partition with $k = 1$ in a number of ways, generalising results from [7] and [21].

Let

$$A^n(G) = \{f \in L^\infty(G^n) : \text{ there exists a normal c.b. multilinear map}$$
$$\Phi : \mathrm{VN}(G)^n \to \mathbb{C} \text{ such that } f(x_n, \dots, x_1) = \Phi(\lambda_{x_n}, \dots, \lambda_{x_1})\}.$$

Since $\{\lambda_x : x \in G\}$ generates $\mathrm{VN}(G)$ as a von Neumann algebra, the element Φ associated with $f \in A^n(G)$ in the above definition is unique. As before, we call f the Fourier transform of Φ and write $f = \hat{\Phi}$. Set $\mathrm{VN}(G)^{\otimes^n_{\sigma h}} = \underbrace{\mathrm{VN}(G) \otimes_{\sigma h} \cdots \otimes_{\sigma h} \mathrm{VN}(G)}_{n}$. By [9], $A^n(G)$ can be identified with the predual of the operator space $\mathrm{VN}(G)^{\otimes^n_{\sigma h}}$ (see [9]). Hence, $A^n(G)$ possesses a canonical operator space structure; up to a complete isometry,

$$A^n(G) \equiv \underbrace{A(G) \otimes_{eh} \cdots \otimes_{eh} A(G)}_{n}.$$

In particular, $\|f\|_{A^n(G)}$ is by definition equal to the completely bounded norm of its associated map Φ. Moreover, the elements $f \in A^n(G)$ have the form

$$f(x_n, \dots, x_1) = \langle \lambda_{x_n} \otimes \cdots \otimes \lambda_{x_1}, f \rangle, \quad x_n, \dots, x_1 \in G.$$

It follows from Corollary 5.7 of [6] that a function $f \in L^\infty(G^n)$ belongs to $A^n(G)$ if and only if there exists an index set J, operators $V_i \in$

$\mathcal{B}(L^2(G)^J)$, $i = 1, \dots, n-1$ and vectors $\xi, \eta \in L^2(G)^J$ such that for all $x_n, \dots, x_1 \in G$ we have

(8)
$$f(x_n, \dots, x_1) = \big((\lambda_{x_n} \otimes 1_J)V_{n-1}(\lambda_{x_{n-1}} \otimes 1_J)V_{n-2} \dots (\lambda_{x_1} \otimes 1_J)\xi, \eta\big).$$

Moreover, $\|f\|_{A^n(G)}$ is equal to the infimum of $\|V_1\| \dots \|V_{n-1}\| \|\xi\| \|\eta\|$ over all representations of the form (8) and this infimum is attained.

A fundamental fact proved by Eymard [10] is that $A(G)$ is an ideal of $B(G)$. We now prove the multidimensional version of this result. In the case G is discrete, this was stated in [7] (p. 214).

Theorem 5.1. *$A^n(G)$ is a closed ideal of $B^n(G)$.*

Proof. We only consider the case $n = 2$; the general case can be treated similarly. Let $f \in A^2(G)$. Then $f(x, y) = ((\lambda_x \otimes 1_J)V(\lambda_y \otimes 1_J)\xi, \eta)$ for some index set J, vectors $\xi, \eta \in L^2(G)^J$ and a bounded operator $V \in \mathcal{B}(L^2(G)^J)$. Letting π be the ampliation of multiplicity J of the left regular representation of $C^*(G)$ on $L^2(G)^J$ and $\Phi \in (C^*(G) \otimes_h C^*(G))^*$ be given by $\Phi(a, b) = (\pi(a)V\pi(b)\xi, \eta)$ we see that $f = \hat{\Phi}$ and hence $f \in B^2(G)$. Thus, $A^2(G) \subseteq B^2(G)$; from the injectivity of the extended Haagerup tensor product it is clear that $A^2(G)$ is closed.

Now let $f \in A^2(G)$ be given as in the first paragraph and $g \in B^2(G)$. By Theorem 4.1, $g(x, y) = (\pi(x)\rho(y)\xi', \eta')$ for some representations $\pi, \rho : G \to H$ and vectors $\xi', \eta' \in H$. Thus,

$$(fg)(x, y) = (((\lambda_x \otimes 1_J \otimes \pi(x)))(V \otimes 1_H)(\lambda_y \otimes 1_J \otimes \rho(y))(\xi \otimes \xi'), \eta \otimes \eta').$$

By [4, Lemma 2.1], there exist unitary operators U and W and index sets J' and J'' such that $U(\lambda_x \otimes 1_J \otimes \pi(x))U^* = \lambda_x \otimes 1_{J'}$ and $W(\lambda_y \otimes 1_J \otimes \rho(y))W^* = \lambda_y \otimes 1_{J''}$. It follows that

$$(fg)(x, y) = (((\lambda_x \otimes 1_{J'})T(\lambda_y \otimes 1_{J''})\xi_0, \eta_0),$$

where $T = U(V \otimes I_H)W^*$, $\xi_0 = W(\xi \otimes \xi')$ and $\eta_0 = U(\eta \otimes \eta')$. This clearly implies that $fg \in A^2(G)$. $\diamondsuit$

Suppose that $1 \le k \le n$. By a block (k, n)-partition we mean a partition of the ordered set $\{n, n-1, \dots, 1\}$ into k subsets of the form $\{\{n, \dots, n_{k-1}\}, \dots, \{n_1 - 1, \dots, 1\}\}$ where $n \ge n_{k-1} > \dots > n_1 > 1$. Suppose that $\mathcal{P}$ is the block (k, n)-partition associated with the sequence $n \ge n_{k-1} > \dots > n_1 > 1$ as above. We define a mapping $\theta_{\mathcal{P}} : A^k(G) \to A^n(G)$ by letting $(\theta_{\mathcal{P}} f)(x_n, \dots, x_1) = f(y_k, \dots, y_1)$ where $y_i = x_{n_i - 1} \dots x_{n_{i-1}}$, $i = 1, \dots, k$, and we have set $n_0 = 1$, $n_k = n + 1$. It follows from (8) that $\theta_{\mathcal{P}}$ maps $A^k(G)$ into $A^n(G)$. We let $\theta = \theta_{\mathcal{P}_0}$ where $\mathcal{P}_0$ is the $(1, n)$-partition; thus, θ maps $A(G)$ into $A^n(G)$.

If $\mathcal{A}$ and $\mathcal{B}$ are algebras and $\mathcal{P}$ is the (k,n)-partition associated with the sequence $n \geq n_{k-1} > \cdots > n_1 > 1$, we say that a map $\Phi : \mathcal{A}^n \to \mathcal{B}$ is $\mathcal{P}$-modular if

$$\Phi(a_n, \dots, a_i a, a_{i-1}, \dots, a_1) = \Phi(a_n, \dots, a_i, a a_{i-1}, \dots, a_1)$$

whenever $a, a_1, \dots, a_n \in \mathcal{A}$ and $i \notin \{1, n_1, \dots, n_{k-1}\}$.

Proposition 5.2. *For each block (k,n)-partition $\mathcal{P}$, the map $\theta_{\mathcal{P}} : A^k(G) \to A^n(G)$ is a completely isometric homomorphism. Moreover,*

$$\operatorname{ran} \theta_{\mathcal{P}} = \{\hat{\Psi} : \Psi : \mathrm{VN}(G)^n \to \mathbb{C} \textit{ is } \mathcal{P}\textit{-modular}\}.$$

Proof. Suppose that $\mathcal{P}$ is associated with the sequence $n \geq n_{k-1} > \cdots > n_1 > 1$. It is obvious that $\theta_{\mathcal{P}}$ is linear and multiplicative. Suppose that $(f_{p,q}) \in M_r(A^k(G))$ and let $\Phi_{p,q} : \mathrm{VN}(G)^k \to \mathbb{C}$ be such that $\hat{\Phi}_{p,q} = f_{p,q}$. Set $\Phi = (\Phi_{p,q})$; then Φ can be viewed as a completely bounded multilinear mapping from $\mathrm{VN}(G)^k$ into M_r. There exist an index set J and operators $V_1, \dots, V_{k-1} \in \mathcal{B}(L^2(G)^J)$, $V_0 : \mathbb{C}^r \to L^2(G)^J$ and $V_k : L^2(G)^J \to \mathbb{C}^r$ such that

$$\Phi(\lambda_{y_k}, \dots, \lambda_{y_1}) = V_k(\lambda_{y_k} \otimes 1_J) V_{k-1} (\lambda_{y_{k-1}} \otimes 1_J) V_{k-2} \dots V_1 (\lambda_{y_1} \otimes 1_J) V_0$$

and $\|\Phi\|_{cb} = \Pi_{i=0}^k \|V_i\|$. Let $\Psi_{p,q} : \mathrm{VN}(G)^n \to \mathbb{C}$ be such that $\hat{\Psi}_{p,q} = \theta_{\mathcal{P}}(f_{p,q})$, $1 \leq p, q \leq r$ and $\Psi = (\Psi_{p,q})$. Then

$$\Psi(\lambda_{x_n}, \dots, \lambda_{x_1}) = V_k(\lambda_{x_n \dots x_{n_{k-1}}} \otimes 1_J) V_{k-1} \dots (\lambda_{x_{n_1 - 1} \dots x_1} \otimes 1_J) V_0. \tag{9}$$

It follows that

$$\|(\theta_{\mathcal{P}}(f_{p,q}))\|_{M_r(A^n(G))} \leq \Pi_{i=0}^k \|V_i\| = \|(f_{p,q})\|_{M_r(A^k(G))},$$

Thus, $\theta_{\mathcal{P}}$ is completely contractive.

Suppose that for some $f \in A^k(G)$ we have $\theta_{\mathcal{P}}(f) = 0$. This implies that $f(x_n \dots x_{n_{k-1}}, \dots, x_{n_1 - 1} \dots x_1) = 0$ for all $x_i \in G$, $i = 1, \dots, n$. Setting $x_i = e$ whenever $i \notin \{1, n_1, \dots, n_{k-1}\}$, we see that $f = 0$. Thus, $\theta_{\mathcal{P}}$ is injective.

Fix $f = (f_{p,q}) \in M_r(A^k(G))$. It is clear from (9) that the element $\Psi = (\Psi_{p,q})$ for which $\hat{\Psi}_{p,q} = \theta_{\mathcal{P}}(f_{p,q})$ is $\mathcal{P}$-modular over $\mathrm{VN}(G)$. By Theorem 2.1,

$$\|\theta_{\mathcal{P}}^{(r)}(f)\|_{M_r(A^n(G))} = \inf \Pi_{i=0}^k \|V_i\|,$$

where the infimum is taken over all operators V_i for which $\Psi(\lambda_{x_n}, \dots, \lambda_{x_1})$ equals the right hand side of (9), for all $x_1, \dots, x_n \in G$. Since θ is injective, if (9) is a representation for Ψ then

$$f(y_k, \dots, y_1) = V_k(\lambda_{y_k} \otimes 1_J) V_{k-1} (\lambda_{y_{k-1}} \otimes 1_J) V_{k-2} \dots (\lambda_{x_1} \otimes 1_J) V_0,$$

for all $y_1, \dots, y_k \in G$. It follows that $\|f\|_{M_r(A^k(G))} \leq \Pi_{i=0}^k \|V_i\|$ and so $\|f\|_{M_r(A^k(G))} \leq \|\theta_{\mathcal{P}}^{(r)}(f)\|_{M_r(A^n(G))}$. Thus, $\theta_{\mathcal{P}}$ is a complete isometry.

Let $\Psi : \mathrm{VN}(G)^n \to \mathbb{C}$ be $\mathcal{P}$-modular. It remains to show that $\hat{\Psi} \in \operatorname{ran}\theta_{\mathcal{P}}$. By Theorem 2.1, there exist an index set and operators $V_1, \dots, V_{k-1}$ and vectors ξ, η such that

$$\Psi(a_n, \dots, a_1) = ((a_n \dots a_{n_k} \otimes 1_J)V_{k-1} \dots V_1(a_{n_1-1} \dots a_1 \otimes 1_J)\xi, \eta),$$

$a_1, \dots, a_n \in \mathrm{VN}(G)$. Letting $f \in A^k(G)$ be the function

$$f(y_k, \dots, y_1) = ((\lambda_{y_k} \otimes 1_J)V_{k-1} \dots V_1(\lambda_{y_1} \otimes 1_J)\xi, \eta),$$

we see that $\theta_{\mathcal{P}}(f) = \hat{\Psi}$. $\diamondsuit$

Definition 5.3. *Let $\mathcal{P}$ be a block (k,n)-partition. We call a function $\varphi \in L^\infty(G^n)$ a $\mathcal{P}$-multiplier of $A(G)$ if*

$$f \in A^k(G) \quad \Rightarrow \quad \varphi\theta_{\mathcal{P}}(f) \in A^n(G).$$

We denote by $M_{\mathcal{P}}A(G)$ the collection of all $\mathcal{P}$-multipliers of $A(G)$.

If $\varphi \in M_{\mathcal{P}}A(G)$ and the map $f \to \varphi\theta_{\mathcal{P}}(f)$ from $A^k(G)$ into $A^n(G)$ is completely bounded we call φ a completely bounded (or c.b.) $\mathcal{P}$-multiplier of $A(G)$. We denote by $M^{cb}_{\mathcal{P}}A(G)$ the collection of all c.b. $\mathcal{P}$-multipliers of $A(G)$.

If $\mathcal{P}$ is the block $(1,n)$-partition we set $M_nA(G) = M_{\mathcal{P}}A(G)$ and $M^{cb}_nA(G) = M^{cb}_{\mathcal{P}}A(G)$.

Remarks (i) If $k = n = 1$ the above definition reduces to that of multipliers and completely bounded multipliers of $A(G)$.

(ii) An application of the Closed Graph Theorem shows that if $\varphi \in M_{\mathcal{P}}A(G)$ then the map $f \to \varphi\theta_{\mathcal{P}}(f)$ from $A^k(G)$ into $A^n(G)$ is bounded.

Proposition 5.4. *Let $\mathcal{P}$ be the block (k,n)-partition associated with the sequence $n \geq n_{k-1} > \dots > n_1 > 1$. The following are equivalent:*
(i) $\varphi \in M^{cb}_{\mathcal{P}}A(G)$;
(ii) The map

$$(\lambda_{x_n}, \dots, \lambda_{x_1}) \to \varphi(x_n, \dots, x_1)\lambda_{x_n \dots x_{n_k}} \otimes \lambda_{x_{n_k-1} \dots x_{n_{k-1}}} \otimes \dots \otimes \lambda_{x_{n_1-1} \dots x_1}$$

extends to a c.b. normal map $\Phi_\varphi : \mathrm{VN}(G)^n \to \mathrm{VN}(G)^{\otimes^k_{\sigma h}}$.

Proof. Suppose that the map $T_\varphi : A^k(G) \to A^n(G)$ given by $f \to \varphi\theta(f)$ is completely bounded. Then its adjoint

$$T^*_\varphi : \mathrm{VN}(G)^{\otimes^n_{\sigma h}} \to \mathrm{VN}(G)^{\otimes^k_{\sigma h}}$$

is completely bounded. For $x_1,\dots,x_n \in G$ set $y_k = x_n \dots x_{n_k},\dots,y_1 = x_{n_1-1}\dots x_1$. If $f \in A(G)$ we have

$$\begin{aligned} & \langle T_\varphi^*(\lambda_{x_n}\otimes\cdots\otimes\lambda_{x_1}), f\rangle = \langle \lambda_{x_n}\otimes\cdots\otimes\lambda_{x_1}, T_\varphi f\rangle \\ = \; & \langle \lambda_{x_n}\otimes\cdots\otimes\lambda_{x_1}, \varphi\theta(f)\rangle = (\varphi\theta(f))(x_n,\dots,x_1) \\ = \; & \varphi(x_n,\dots,x_1) f(y_k,\dots,y_1) = \langle \varphi(x_n,\dots,x_1)\lambda_{y_k}\otimes\cdots\otimes\lambda_{y_1}, f\rangle. \end{aligned}$$

Thus, the map Φ_φ in (ii) can be taken to be T_φ^*. Conversely, if (ii) holds then the map Φ_φ in (ii) has a completely bounded predual T_φ and the chain of equalities above implies (i). ♢

The mapping $\varphi \to \Phi_\varphi$ from Proposition 5.4 is an embedding of $M_{\mathcal{P}}^{cb}A(G)$ into the space of all normal completely bounded maps from $\mathrm{VN}(G)^{\otimes_{\sigma h}^n}$ into $\mathrm{VN}(G)^{\otimes_{\sigma h}^k}$ and hence gives rise to an operator space structure on $M_{\mathcal{P}}^{cb}A(G)$. Namely, given a matrix

$$\varphi = (\varphi_{p,q}) \in M_m(M_{\mathcal{P}}^{cb}A(G))$$

we let $\|\varphi\|_{M_m(M_{\mathcal{P}}^{cb}A(G))} = \|\Phi_\varphi\|_{cb}$, where $\Phi_\varphi \stackrel{def}{=} (\Phi_{\varphi_{p,q}})$ is the corresponding mapping from $\mathrm{VN}(G)^{\otimes_{\sigma h}^n}$ into $M_m(\mathrm{VN}(G)^{\otimes_{\sigma h}^k})$.

In the next theorem, we relate the completely bounded $\mathcal{P}$-multipliers to multidimensional Schur multipliers in the case where $\mathcal{P}$ is the $(1,n)$-partition. It generalises Theorem 4.1 of [7], which concerns discrete groups, to arbitrary locally compact groups.

Theorem 5.5. *Let $\varphi \in L^\infty(G^n)$ and $\mathcal{S}$ be the space of all $n+1$-dimensional Schur multipliers with respect to the left Haar measure on G. The following are equivalent:*

(i) $\varphi \in M_n^{cb}A(G)$;

(ii) The function $\tilde\varphi \in L^\infty(G^{n+1})$ given by

$$\tilde\varphi(x_1,\dots,x_{n+1}) = \varphi(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)$$

belongs to $\mathcal{S}$.

Moreover, if $k \in \mathbb{N}$ and $\varphi_{p,q} \in M_n^{cb}A(G)$, $1 \le p,q \le k$, then

$$\|(\varphi_{p,q})\|_{M_k(M_n^{cb}A(G))} = \|(\tilde\varphi_{p,q})\|_{M_k(\mathcal{S})}.$$

Proof. (i)⇒(ii) Let $\varphi = (\varphi_{p,q}) \in M_k(M_n^{cb}A(G))$ with $\|\varphi\|_{M_k(M_n^{\mathrm{cb}}A(G))} < 1$, $\Phi_{\varphi_{p,q}}$ be the c.b. normal map from Proposition 5.4, and $\Phi_\varphi = (\Phi_{\varphi_{p,q}})$. By [6], there exist operators $V_i \in \mathcal{B}(L^2(G)^\infty)$, $i = 2,\dots,n$, $V_1 \in \mathcal{B}(L^2(G)^k, L^2(G)^\infty)$ and $V_{n+1} \in \mathcal{B}(L^2(G)^\infty, L^2(G)^k)$ such that $\Pi_{i=1}^{n+1}\|V_i\| < 1$ and

$$(\varphi_{p,q}(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)\lambda_{x_{n+1}^{-1}}\lambda_{x_1})_{p,q} =$$

$$V_{n+1}(\lambda_{x_{n+1}^{-1}}\lambda_{x_n}\otimes 1)V_n(\lambda_{x_n^{-1}}\lambda_{x_{n-1}}\otimes 1)V_{n-1}\dots(\lambda_{x_2^{-1}}\lambda_{x_1}\otimes 1)V_1, \tag{10}$$

where the ampliations are of infinite countable multiplicity. Let $a_1 : G \to \mathcal{B}(L^2(G)^k, L^2(G)^\infty)$ and $a_{n+1} : G \to \mathcal{B}(L^2(G)^\infty, L^2(G)^k)$ be given as follows:

$$a_1(x_1) = (\lambda_{x_1}\otimes 1)V_1(\lambda_{x_1^{-1}}\otimes 1_k), \;\; a_{n+1}(x_{n+1}) = (\lambda_{x_{n+1}}\otimes 1_k)V_{n+1}(\lambda_{x_{n+1}^{-1}}\otimes 1).$$

Let also $a_i : G \to \mathcal{B}(L^2(G)^\infty)$, $i = 2, \dots, n$, be given by

$$a_i(x_i) = (\lambda_{x_i} \otimes 1)V_i(\lambda_{x_i^{-1}} \otimes 1), \quad x_i \in G.$$

It follows from (10) that, for all $x_1, \dots, x_{n+1}$, we have

$$\begin{aligned} & \varphi(x_{n+1}^{-1}x_n, \dots, x_2^{-1}x_1) \otimes 1_{L^2(G)} \\ = \;& (\varphi_{p,q}(x_{n+1}^{-1}x_n, \dots, x_2^{-1}x_1)1_{L^2(G)})_{p,q} \\ = \;& (\lambda_{x_{n+1}} \otimes 1_k)(\varphi_{p,q}(x_{n+1}^{-1}x_n, \dots, x_2^{-1}x_1)\lambda_{x_{n+1}^{-1}}\lambda_{x_1})_{p,q}(\lambda_{x_1^{-1}} \otimes 1_k) \\ = \;& a_{n+1}(x_{n+1})a_n(x_n)\dots a_1(x_1). \end{aligned}$$

Let ξ be a unit vector in $L^2(G)$ and E be the projection onto the one dimensional subspace of $L^2(G)$ generated by ξ. The last identity implies that $\varphi(x_{n+1}^{-1}x_n, \dots, x_2^{-1}x_1) = (Ea_{n+1}(x_{n+1}))a_n(x_n)\dots a_2(x_2)(a_1(x_1)E)$, for all $x_i \in G$, $i = 1, \dots, n+1$. It follows from Theorem 3.2 that $\tilde{\varphi}_{p,q} \in \mathcal{S}$ and

$$\|(\tilde{\varphi}_{p,q})\|_{\mathrm{m},k} \leq \Pi_{i=1}^{n+1}\|V_i\| < 1.$$

(ii)⇒(i) Let $\varphi \in L^\infty(G^n)$ and suppose that $\tilde{\varphi}$ is a Schur multiplier with respect to the left Haar measure. By Theorem 3.4 of [15], the function $\psi \in L^\infty(G^{n+1})$ given by $\psi(y_1, \dots, y_{n+1}) = \tilde{\varphi}(y_1^{-1}, \dots, y_{n+1}^{-1})$, $y_1, \dots, y_{n+1} \in G$, is also a Schur multiplier with respect to the left Haar measure. Set $y_i = x_i^{-1}x_{i+1}^{-1}\dots x_n^{-1}s$, $i = 1, \dots, n$, and $y_{n+1} = s$. We have that

$$\psi(y_1, \dots, y_{n+1}) = \varphi(y_{n+1}y_n^{-1}, y_ny_{n-1}^{-1}, \dots, y_2y_1^{-1}) = \varphi(x_n, x_{n-1}, \dots, x_1).$$

By Theorem 3.4 of [15], there exist functions $a_i : G \to M_\infty$, $i = 2, \dots, n$, $a_1 : G \to M_{\infty,1}$ and $a_{n+1} : G \to M_{1,\infty}$ such that

$$\psi(y_1, \dots, y_{n+1}) = a_{n+1}(y_{n+1})a_n(y_n)\dots a_1(y_1), \quad y_1, \dots, y_{n+1} \in G.$$

For each $i = 2, \dots, n$, let $A_i \in \mathcal{B}(L^2(G) \otimes \ell^2)$ be the operator corresponding in a canonical way to a_i. Namely, A_i is given by $(A_i\tilde{\xi})(s) = a_i(s)\tilde{\xi}(s)$, $s \in G$, where we have identified $L^2(G) \otimes \ell^2$ with the space $L^2(G; \ell_2)$ of all square integrable ℓ^2-valued functions on G. Similarly, let $A_1 \in \mathcal{B}(L^2(G), L^2(G) \otimes \ell^2)$ and $A_{n+1} \in \mathcal{B}(L^2(G) \otimes \ell^2, L^2(G))$ be the operators corresponding to a_1 and a_{n+1}, respectively.

Let $f \in A(G)$. Then there exist $\xi, \eta \in L^2(G)$ such that

$$\theta(f)(x_n, \dots, x_1) = (\lambda_{x_n \dots x_1}\xi, \eta) = \int_G \xi(x_1^{-1} \dots x_n^{-1}s)\overline{\eta(s)}dm(s).$$

We have

$$\begin{aligned}
& (\varphi\theta(f))(x_n, \dots, x_1) \\
= \ & \varphi(x_n, \dots, x_1) f(x_n \dots x_1) \\
= \ & \int_G \varphi(x_n, \dots, x_1)\xi(x_1^{-1} \dots x_n^{-1}s)\overline{\eta(s)}dm(s) \\
= \ & \int_G \psi(x_1^{-1} \dots x_n^{-1}s, \dots, x_n^{-1}s, s)\xi(x_1^{-1} \dots x_n^{-1}s)\overline{\eta(s)}dm(s) \\
= \ & \int_G a_{n+1}(s)a_n(x_n^{-1}s) \dots a_1(x_1^{-1} \dots x_n^{-1}s)\xi(x_1^{-1} \dots x_n^{-1}s)\overline{\eta(s)}dm(s) \ .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (A_{n+1}(\lambda_{x_n} \otimes 1)A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi, \eta) \\
= \ & \big((\lambda_{x_n} \otimes 1)A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi, A_{n+1}^*\eta\big) \\
= \ & \int_G \big(((\lambda_{x_n} \otimes 1)A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi)(s), (A_{n+1}^*\eta)(s)\big)_{\ell_2} dm(s) \\
= \ & \int_G a_{n+1}(s)((\lambda_{x_n} \otimes 1)A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi)(s)\overline{\eta(s)}dm(s) \\
= \ & \int_G a_{n+1}(s)(A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi)(x_n^{-1}s)\overline{\eta(s)}dm(s) \\
= \ & \int_G a_{n+1}(s)a_n(x_n^{-1}s)(\lambda_{x_{n-1}} \otimes 1) \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi)(x_n^{-1}s)\overline{\eta(s)}dm(s) \\
= \ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
= \ & \int_G a_{n+1}(s)a_n(x_n^{-1}s) \dots a_1(x_1^{-1} \dots x_n^{-1}s)\xi(x_1^{-1} \dots x_n^{-1}s)\overline{\eta(s)}dm(s) \ .
\end{aligned}$$

It follows that

$$(\varphi\theta(f))(x_n, \dots, x_1) = (A_{n+1}(\lambda_{x_n} \otimes 1)A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1\xi, \eta) \tag{11}$$

and hence $\varphi\theta(f) \in A^n(G)$. Thus, $\varphi \in M_nA(G)$ and, by Remark (ii) after Definition 5.3, the map $f \to \varphi\theta(f)$ is bounded. Equation (11) implies that if Φ_φ is its adjoint then

$$\Phi_\varphi(\lambda_{x_n} \otimes \dots \otimes \lambda_{x_1}) = A_{n+1}(\lambda_{x_n} \otimes 1) \dots (\lambda_{x_1} \otimes 1)A_1, \quad x_1, \dots, x_n \in G. \tag{12}$$

Thus, Φ_φ is completely bounded, and hence $\varphi \in M_n^{cb}A(G)$.

Now suppose that $\varphi = (\varphi_{p,q}) \in M_k(L^\infty(G^n))$ and that $\|(\tilde{\varphi}_{p,q})\|_{\mathrm{m},k} < 1$. Let $\psi_{p,q}$ be the map corresponding to $\varphi_{p,q}$ as specified in the case

$k = 1$ above and $\psi = (\psi_{p,q})$. Theorem 3.2 implies that $\|\psi\|_{\mathrm{m},k} = \|\tilde{\varphi}\|_{\mathrm{m},k} < 1$. Thus, in the notation of Theorem 3.2, $\|\tilde{S}_\psi\|_k < 1$, where $\tilde{S}_\psi = (\tilde{S}_{\psi_{p,q}})_{p,q}$ is the canonical normal completely bounded multilinear map from $\mathcal{B}(L^2(G)) \times \cdots \times \mathcal{B}(L^2(G))$ into $M_k(\mathcal{B}(L^2(G)))$. By Theorem 3.2, we can write $\psi(y_1, \dots, y_{n+1}) = a_{n+1}(y_{n+1}) \dots a_1(y_1)$, where $a_i : G \to M_\infty$, $i = 2, \dots, n$, $a_1 : G \to M_{\infty,k}$ and $a_{n+1} : G \to M_{k,\infty}$ are functions such that $\operatorname{esssup}_{y_1,\dots,y_{n+1}\in G} \Pi_{i=1}^{n+1} \|a_i(y_i)\| < 1$. As before, let $A_i \in \mathcal{B}(L^2(G)^\infty)$, $i = 2, \dots, n$, $A_1 \in \mathcal{B}(L^2(G)^k, L^2(G)^\infty)$ and $A_{n+1} \in \mathcal{B}(L^2(G)^\infty, L^2(G)^k)$ be the operators corresponding to the a_i's in the canonical way. Let A_{n+1}^p (resp. A_1^q) be the pth row (resp. the qth column) of A_{n+1} (resp. A_1). By (12), $\Phi_{\varphi_{p,q}}(\lambda_{x_n} \otimes \cdots \otimes \lambda_{x_1}) = A_{n+1}^p(\lambda_{x_n} \otimes 1)A_n \dots A_2(\lambda_{x_1} \otimes 1)A_1^q$, for all $x_1, \dots, x_n \in G$. It follows that if $\Phi_\varphi = (\Phi_{\varphi_{p,q}})$ then (12) holds in the case under consideration as well. Since $\Pi_{i=1}^{n+1}\|A_i\| < 1$, we conclude that $\|\Phi_\varphi\|_{cb} < 1$ or, equivalently, $\|\varphi\|_{M_k(M_n^{cb}A(G))} < 1$. $\diamondsuit$

Corollary 5.6. *We have that $B^n(G) \subset M_n^{cb}A(G)$. Moreover, the inclusion map is a complete contraction.*

Proof. The inclusion follows from Theorem 4.1, Theorem 5.5 and Theorem 3.4 of [15].

Let $\varphi = (\varphi_{p,q}) \in M_k(B^n(G))$, $\|\varphi\|_{M_k(B^n(G))} < 1$ and $\Phi : C^*(G)^n \to M_k$ be the completely bounded mapping associated with φ. By Theorem 5.2 of [6], there exist Hilbert spaces $H_1, \dots, H_n$, representations $\pi_i : C^*(G) \to \mathcal{B}(H_i)$ and operators $V_1 \in \mathcal{B}(H, \mathbb{C}^k)$, $V_{n+1} \in \mathcal{B}(\mathbb{C}^k, H)$ and $V_i \in \mathcal{B}(H)$, $i = 2, \dots, n$, such that

$$\Phi(a_1, \dots, a_n) = V_1\pi_1(a_1)V_2 \dots V_n\pi_n(a_n)V_{n+1}$$

and $\Pi_{i=1}^{n+1}\|V_i\| < 1$. Let $\tilde{\pi}_i : W^*(G) \to \mathcal{B}(H)$ be the canonical normal extension of π_i, $i = 1, \dots, n$. Since the extension $\tilde{\Phi}$ of Φ to a normal completely bounded map from $W^*(G)^n$ into M_k is unique, we have that

$$\tilde{\Phi}(b_1, \dots, b_n) = V_1\tilde{\pi}_1(b_1)V_2 \dots V_n\tilde{\pi}_n(b_n)V_{n+1}, \quad b_1, \dots, b_n \in W^*(G).$$

Let $a_1(y_1) = \tilde{\pi}_n(\omega(y_1))V_{n+1}$, $a_2(y_2) = \tilde{\pi}_{n-1}(\omega(y_2))V_n\tilde{\pi}_n(\omega(y_2^{-1}))$, $\dots$, $a_{n+1}\ (y_{n+1}) = V_1\tilde{\pi}_1(\omega(y_{n+1}^{-1}))$. Then

$$\begin{aligned} \tilde{\varphi}(y_1, \dots, y_{n+1}) &= \tilde{\Phi}(\omega(y_{n+1}^{-1})\omega(y_n), \dots, \omega(y_2^{-1})\omega(y_1)) \\ &= a_{n+1}(y_{n+1}) \dots a_1(y_1) \end{aligned}$$

and $\operatorname{esssup}_{y_1,\dots,y_{n+1}\in G} \Pi_{i=1}^{n+1}\|a_i(y_i)\| < 1$. Theorems 3.2 and 5.5 imply that the norm of φ as an element of $M_k(M_n^{cb}A^n(G))$ is less than one. Thus, the inclusion $B^n(G) \subset M_n^{cb}A(G)$ is a complete contraction. $\diamondsuit$

We recall that $C_r^*(G)$ is the reduced C*-algebra of G. We write $C_r^*(G)^{\otimes_h^n}$ for $\underbrace{C_r^*(G) \otimes_h \ldots \otimes_h C_r^*(G)}_{n}$. Let $B_r(G) = C_r^*(G)^*$ and $B_r^n(G) = (C_r^*(G)^{\otimes_h^n})^*$. It is standard to identify the elements of $B_r(G)$ with functions from $B(G)$ in such a way that the duality between $B_r(G)$ and $C_r^*(G)$ is given by $\langle b, \lambda(f)\rangle = \int f(x)b(x)dm(x)$, $f \in L^1(G)$. We equip $B_r(G)$ and $B_r^n(G)$ with the canonical operator space structure as dual operator spaces. Let M be the completely contractive mapping from $C_r^*(G)^{\otimes_h^n}$ to $C_r^*(G)$ which maps $\lambda(f_1) \otimes \ldots \otimes \lambda(f_n)$ (for $f_1, \ldots, f_n \in L^1(G)$) to $\lambda(f)$, where

$$f(x) = \int_{G^n} f_1(x_1)f_2(x_1^{-1}x_2)\ldots f_n(x_{n-1}^{-1}x)dm(x_1)\ldots dm(x_{n-1}).$$

It is easy to check that the adjoint mapping M^* maps $f \in B_r(G)$ to $\theta(f) \in B_r^n(G)$ (here $\theta(f)(x_1, \ldots, x_n) = f(x_1 \ldots x_n)$). We define $M_n^{cb}B_r(G)$ to be the space of all $\varphi \in L^\infty(G^n)$ such that the mapping $T_\varphi : f \mapsto \varphi\theta(f)$ is completely bounded as a map from $B_r(G)$ to $B_r^n(G)$. We note that this map is normal. In fact, if $f_1, \ldots, f_n \in L^1(G)$ then

$$\begin{aligned} & \langle \varphi\theta(f), \lambda(f_1) \otimes \ldots \otimes \lambda(f_n)\rangle \\ = & \int_{G^n} \varphi(x_1, \ldots, x_n)f(x_1 \ldots x_n)f_1(x_1)\ldots f_n(x_n)dm(x_1)\ldots dm(x_n) \\ = & \langle f, \lambda(g)\rangle, \end{aligned}$$

where $g(x)$ equals

$$\int f_1(x_1)f_2(x_1^{-1}x_2)\ldots f_n(x_{n-1}^{-1}x)\varphi(x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x)dm(x_1)\ldots dm(x_{n-1});$$

it is easy to see that $g \in L^1(G)$. Therefore T_φ has a predual M_φ which is given by $\lambda(f_1) \otimes \ldots \otimes \lambda(f_n) \mapsto \lambda(g)$. If $\varphi \in M_n^{cb}B_r(G)$ then M_φ is completely bounded and $\|\varphi\|_{M_n^{cb}B_r(G)} = \|M_\varphi\|_{cb}$. From the definition of the operator space structure of $B_r(G)$, we have that if $(\varphi_{p,q}) \in M_k(M_{cb}^n B_r(G))$ then $\|(\varphi_{p,q})\| = \|M_\varphi\|_{cb}$, where $M_\varphi = (M_{\varphi_{p,q}})$ is the corresponding mapping from $C_r^*(G)^{\otimes_h^n}$ to $M_k(C_r^*(G))$.

The following theorem supplements Theorem 5.5 and provides a multidimensional version of Proposition 4.1 of [21].

Theorem 5.7. *Let $\varphi \in M_k(L^\infty(G^n))$. Then the following are equivalent*

(i) $\varphi \in b_1(M_k(M_n^{cb}A(G)))$;

(ii) the multilinear mapping $M_\varphi : (\lambda(f_1), \dots, \lambda(f_n)) \mapsto (\lambda(f_{ij}))$, *where* $f_1, \dots, f_n \in L^1(G)$ *and* $f_{ij}(x)$ *equals*

$$\int f_1(x_1)f_2(x_1^{-1}x_2)\dots f_n(x_{n-1}^{-1}x)\varphi_{ij}(x_1, x_1^{-1}x_2, \dots, x_{n-1}^{-1}x)dm(x_1)\dots dm(x_{n-1})$$

extends to a complete contraction from $C_r^*(G)^{\otimes_h^n}$ *into* $M_k(C_r^*(G))$;

(iii) $\varphi \in b_1(M_k(M_n^{cb}B_r(G)))$.

Proof. For the sake of technical simplicity we assume that $n = 2$; the general case can be treated similarly.

(i)⇒(ii) Let $\varphi = (\varphi_{p,q}) \in b_1(M_k(M_2^{cb}A(G)))$. By Proposition 5.4, there exist operators $V_0 \in \mathcal{B}(L^2(G)^k, L^2(G)^\infty)$, $V_1 \in \mathcal{B}(L^2(G)^\infty)$ and $V_2 \in \mathcal{B}(L^2(G)^\infty, L^2(G)^k)$ such that $\|V_0\|\|V_1\|\|V_2\| \leq 1$ and

$$\varphi(x_2, x_1)\lambda_{x_2x_1} = V_2(\lambda_{x_2} \otimes 1)V_1(\lambda_{x_1} \otimes 1)V_0. \tag{13}$$

Let $f_1 = (f_1^{p,q}) \in M_{k,r}(C_r^*(G))$ and $f_2 = (f_2^{p,q}) \in M_{r,k}(C_r^*(G))$. We denote by $\lambda(f_1) \odot \lambda(f_2) \in M_k(C_r^*(G) \otimes_h C_r^*(G))$ a $k \times k$-matrix whose (p, q) entry equals $\sum_{s=1}^r \lambda(f_{p,s}^1) \otimes \lambda(f_{s,q}^2)$. If $f_{p,q}^l \in L^1(G)$, $l = 1, 2$, then

$$\begin{aligned}
& M_\varphi^{(k)}(\lambda(f_1) \odot \lambda(f_2)) \\
= & \left(\sum_{s=1}^r \int f_{p,s}^1(x_1)f_{s,q}^2(x_1^{-1}x_2)\varphi(x_1, x_1^{-1}x_2)\lambda(x_2)dm(x_1)dm(x_2)\right)_{p,q} \\
= & \left(\sum_{s=1}^r \int f_{p,s}^1(x_1)f_{s,q}^2(x_2)\varphi(x_1, x_2)\lambda(x_1x_2)dm(x_1)dm(x_2)\right)_{p,q} \\
= & \left(\int \sum_{s=1}^r f_{p,s}^1(x_1)f_{s,q}^2(x_2)V_2(\lambda_{x_1} \otimes 1)V_1(\lambda_{x_2} \otimes 1)V_0 dm(x_1)dm(x_2)\right)_{p,q} \\
= & \left(\sum_{s=1}^r V_2((\int f_{p,s}^1(x_1)\lambda_{x_1}dm(x_1)) \otimes 1)V_1((\int f_{s,q}^2(x_2)\lambda_{x_2}dm(x_2)) \otimes 1)V_0\right)_{p,q} \\
= & \left(\sum_{s=1}^r V_2(\lambda(f_{p,s}^1) \otimes 1)V_1(\lambda(f_{s,q}^2) \otimes 1)V_0\right)_{p,q}.
\end{aligned}$$

Therefore

$$\|M_\varphi^{(k)}(\lambda(f_1) \odot \lambda(f_2)\| \leq \|V_0\|\|V_1\|\|V_2\|\|\lambda(f_1)\|\|\lambda(f_2)\|$$

and hence $\|M_\varphi^{(k)}\| \leq 1$.

(ii)⇔(iii) Follows trivially from the definition of the operator structure of $M_n^{cb}B_r(G)$.

(iii)⇒(i) We only consider the case $k = 1$. Let $\varphi \in M_n^{cb} B_r(G)$, $\|\varphi\| \leq 1$ and $\psi \in A(G) \cap C_c(G)$, where $C_c(G)$ is the space of compactly supported functions on G. We can find $g \in A(G)$ such that $g = 1$ on the support of ψ so that $\psi g = \psi$. As $\theta(g) \in A^n(G)$ and $A^n(G)$ is an ideal in $B_r^n(G)$ we have $\varphi\theta(\psi) = \varphi\theta(\psi)\theta(g) \in A^n(G)$. Since the $A^n(G)$-norm and $B_r^n(G)$-norm coincide on $A^n(G)$ and $A(G) \cap C_c(G)$ is dense in $A(G)$ we obtain that φ is in $b_1(M_n(G))$. Similar arguments show that φ is a completely contractive multiplier. $\diamondsuit$

We next supply some corollaries of the previous results.

Corollary 5.8. *Let G be an amenable locally compact group. Then $B^n(G) = M_n^{cb} A(G)$ completely isometrically.*

Proof. If G is amenable then $B^n(G) = B_r^n(G)$ completely isometrically. Hence, by Theorem 5.7, $M_n^{cb} A(G) = M_n^{cb} B(G)$ completely isometrically. Since $B(G)$ contains the constant functions, it is easy to see that $M_n^{cb} B(G) = B^n(G)$ completely isometrically. $\diamondsuit$

Corollary 5.9. *Let $\mathcal{P}$ be the block (k, n)-partition associated with the sequence $n \geq n_k > \cdots > n_1 > 1$ such that each block contains at least two elements, and $\epsilon_i = \pm 1$, $i = 1, \ldots, n$. Assume that G is amenable. Then the function $\psi : G^n \to \mathbb{C}$ given by $\psi(s_n, \ldots, s_1) = \varphi(s_1^{\varepsilon_1} \ldots s_{n_1-1}^{\varepsilon_{n_1-1}}, \ldots, s_{n_{k-1}}^{\varepsilon_{n_{k-1}}} \ldots s_n^{\varepsilon_n})$ is a Schur multiplier with respect to the left Haar measure if and only if $\varphi \in B^k(G)$.*

Proof. We prove the statement for $k = 2$ and a partition of the form $\mathcal{P} = \{\{n, \ldots, m\}, \{m-1, \ldots, 1\}\}$; the other cases are similar. Assume ψ is a Schur multiplier. Then $\psi(s_n, \ldots, s_1) = a_1(s_1) \ldots a_n(s_n)$ for some (essentially bounded) functions $a_i : G \to M_\infty$, $i = 2, \ldots, n-1$, $a_n : G \to M_{\infty,1}$ and $a_1 : G \to M_{1,\infty}$. Therefore, the function

$$(s_1, s_2, s_3) \mapsto \varphi(s_3^{-1} s_2, s_2^{-1} s_1) = \psi(s_1^{\varepsilon_n}, s_2^{-\varepsilon_{n-1}}, e, \ldots, e, s_2^{\varepsilon_2}, s_3^{-\varepsilon_1})$$

is a Schur multiplier and hence by Theorem 5.5, $\varphi \in M_2^{cb} A(G) = B^2(G)$.

Let now $\varphi \in B^2(G)$. By Theorem 4.1, there exist representations π_1, π_2 of G on H and vectors ξ, η such that $\varphi(s_2, s_1) = (\pi_2(s_2)\pi_1(s_1)\xi, \eta)$, and

$$\psi(s_n, \ldots, s_1) = (\pi_2(s_1^{\varepsilon_1} \ldots s_{m-1}^{\varepsilon_{m-1}})\pi_1(s_m^{\varepsilon_m} \ldots s_n^{\varepsilon_n})\xi, \eta).$$

Theorem 3.4 of [15] now easily implies ψ is a Schur multiplier. $\diamondsuit$

Remark 5.10. Since if G is abelian then $B(G) = \{\hat{\mu} : \mu \in M(\hat{G})\}$, Corollary 5.9 implies the following classical result: If G is a discrete abelian group and $\varphi \in l^\infty(G)$ then the function ψ given by $\psi(x,y) = \varphi(y^{-1}x)$ is a Schur multiplier if and only if $\varphi = \hat{\mu}$ for some measure $\mu \in M(\hat{G})$.

Here is a more general result:

Corollary 5.11. *Let G be a locally compact abelian group, $m_1, \dots, m_n = \pm 1$, $\varphi \in L^\infty(G)$ and ψ be the function given by*

$$\psi(s_n, \dots, s_1) = \varphi(s_1^{m_1} \dots s_n^{m_n}), \quad s_1, \dots, s_n \in G.$$

Then ψ is a Schur multiplier (with respect to the Haar measure) if and only if $\varphi = \hat{\mu}$ for some measure $\mu \in M(\hat{G})$. In this case, $\|\psi\|_{\rm m} = \|\mu\|$.

We close this section with a multidimensional version of [5, Theorem 1]. We use the notation from Proposition 5.4. Recall [10] that if $f \in A(G)$ and $T \in \mathrm{VN}(G)$ then $fT \in \mathrm{VN}(G)$ is the operator given by the duality relation $\langle g, fT\rangle = \langle fg, T\rangle$.

Proposition 5.12. *Let $\Phi : \mathrm{VN}(G)^n \to \mathrm{VN}(G)$ be a normal completely bounded multilinear map. Then $\Phi = \Phi_\varphi$ for some $\varphi \in M_n^{cb}A(G)$ if and only if*

$$\Phi(\theta(f)(S_1 \otimes \dots \otimes S_n)) = f\Phi(S_1 \otimes \dots \otimes S_n), \tag{14}$$

for all $f \in A(G)$ and all $S_1, \dots, S_n \in \mathrm{VN}(G)$.

Proof. Since Φ is a normal completely bounded map, $\Phi = \Psi^*$ for a completely bounded map from $A(G)$ to $A^n(G)$,

$$\langle \Phi(\theta(f)(S_1 \otimes \dots \otimes S_n)), h\rangle = \langle S_1 \otimes \dots \otimes S_n, \theta(f)\Psi(h)\rangle$$

and

$$\langle f\Phi(S_1 \otimes \dots \otimes S_n), h\rangle = \langle S_1 \otimes \dots \otimes S_n, \Psi(fh)\rangle$$

Thus, if Φ satisfies (14) then $\theta(f)\Psi(h) = \Psi(fh)$ for all f, $h \in A(G)$. Since $A(G)$ is commutative, $\theta(f)\Psi(h) = \theta(h)\Psi(f)$ and therefore $\Psi(h) = \varphi\theta(h)$ for some function φ on G^n. Since Ψ is completely bounded, $\varphi \in M_n^{cb}A(G)$. Moreover,

$$\langle \Phi(\lambda_{x_n} \otimes \dots \otimes \lambda_{x_1}), h\rangle = \langle \lambda_{x_n} \otimes \dots \otimes \lambda_{x_1}, \varphi\theta(h)\rangle = \\ \varphi(x_n, \dots, x_1)h(x_n \dots x_1) = \langle \varphi(x_n, \dots, x_1)\lambda_{x_n \dots x_1}, h\rangle,$$

that is, $\Phi = \Phi_\varphi$. $\diamondsuit$

6. The abelian case

In this section we assume that G is abelian. We denote by $\hat{G}$ the character group of G. Let $C_0(G)$ be the algebra of continuous functions vanishing at infinity on G. The Haagerup tensor product $\underbrace{C_0(G) \otimes_{\mathrm{h}} \ldots \otimes_{\mathrm{h}} C_0(G)}_{n}$ will be denoted by $V_{\mathrm{h}}^n(G)$. The dual space of $V_{\mathrm{h}}^n(G)$ is the space of n-measures on $\hat{G}$. Let $C_b(G)$ be the C^*-algebra of continuous bounded functions on G and $\mathcal{V}^n(G) = \underbrace{C_b(G) \otimes_{\mathrm{h}} \ldots \otimes_{\mathrm{h}} C_b(G)}_{n}$.

Denote by $\hat{G}_d$ the group $\hat{G}$ equipped with the discrete topology and recall that the Bohr compactification $\bar{G}$ of G is the dual of $\hat{G}_d$. We note that there is a canonical inclusion of $V_{\mathrm{h}}^n(G)^*$ into $V_{\mathrm{h}}^n(\bar{G})^*$: for $\Phi \in V_{\mathrm{h}}^n(G)^*$ define $\bar{\Phi} \in V_{\mathrm{h}}^n(\bar{G})^*$ by

$$\bar{\Phi}(a_1 \otimes \cdots \otimes a_n) = \tilde{\Phi}(\iota(a_1|_G) \otimes \cdots \otimes \iota(a_n|_G)), \quad a_1, \ldots, a_n \in C(\bar{G}),$$

where $\tilde{\Phi}$ is the extension of Φ to a normal completely bounded multilinear map from $(C_0(G)^{**})^{\otimes^n_{\sigma h}}$ to $\mathbb{C}$, and $\iota : C_b(G) \to C_0(G)^{**}$ is the canonical injection.

We claim that

$$\|\bar{\Phi}\|_{V_{\mathrm{h}}^n(\bar{G})^*} = \|\Phi\|_{V_{\mathrm{h}}^n(G)^*}. \tag{15}$$

If $a_k = (a^k_{i,j})$, $k = 1, \ldots, n$, are n by n matrices let $a_1 \odot \cdots \odot a_n$ be the n by n matrix whose (i,j)-entry is equal to

$$a^1_{i,i_1} \otimes a^2_{i_1,i_2} \otimes \cdots \otimes a^n_{i_{n-1},j}.$$

To show (15), first note that if $a_1 \odot \ldots \odot a_n \in V_{\mathrm{h}}^n(\bar{G})$ is a function of unit Haagerup norm then

$$|\bar{\Phi}(a_1 \odot \ldots \odot a_n)| = |\tilde{\Phi}(\iota(a_1|_G) \odot \ldots \odot \iota(a_n|_G))| \leq \|\Phi\|,$$

where for $a = (a_{ij}) \in M_{k,l}(C(\bar{G}))$ we denote by $a|_G$ the matrix $(a_{ij}|_G)$. Hence, $\|\bar{\Phi}\|_{V_{\mathrm{h}}^n(\bar{G})^*} \leq \|\Phi\|_{V_{\mathrm{h}}^n(G)^*}$. Conversely, let $\bar{a}$ denote the canonical extension of a function a from $C_0(G)$ to a function from $C(\bar{G})$ and $\bar{u} \in V_{\mathrm{h}}^n(\bar{G})$ denote the corresponding extension of an element $u \in V_{\mathrm{h}}^n(G)$. Thus, if $u = a_1 \odot \cdots \odot a_n$ then $\bar{u} = \bar{a}_1 \odot \cdots \odot \bar{a}_n$. It follows that $\|\bar{u}\|_{V_{\mathrm{h}}^n(\bar{G})} \leq \|u\|_{V_{\mathrm{h}}^n(G)}$ and hence

$$\begin{aligned}
\|\Phi\|_{V_{\mathrm{h}}^n(G)^*} &= \sup\{|\Phi(u)| : u \in V_{\mathrm{h}}^n(G), \|u\|_{\mathrm{h}} \leq 1\} \\
&= \sup\{|\bar{\Phi}(\bar{u})| : u \in V_{\mathrm{h}}^n(G), \|u\|_{\mathrm{h}} \leq 1\} \\
&\leq \sup\{|\bar{\Phi}(v)| : v \in V_{\mathrm{h}}^n(\bar{G}), \|v\|_{\mathrm{h}} \leq 1\} \\
&= \|\bar{\Phi}\|_{V_{\mathrm{h}}^n(\bar{G})^*}.
\end{aligned}$$

Thus (15) is established. We hence have a canonical isometric embedding of $M^n(\hat{G})$ into $M^n(\hat{G}_d)$, which gives rise to an isometric embedding of $B^n(\hat{G})$ into $B^n(\hat{G}_d)$. The next proposition generalises [12, Theorem 3.3] to the multidimensional case. We note that the proof we give is new in the case $n = 2$ as well.

Proposition 6.1. *Let $f \in B^n(\hat{G}_d)$. Then $f \in B^n(\hat{G})$ if and only if f is continuous.*

Proof. It is clear that if $f \in B^n(\hat{G})$ then f is continuous. For the converse direction we use induction on n. If $n = 1$ the claim follows from a classical result of Eberlein [20, Theorem 1.9.1]. Suppose that $n > 1$ and fix a continuous function f from $B^n(\hat{G}_d)$. For an element $\gamma \in \hat{G}$ let $\delta_\gamma \in B(\hat{G}_d)^*$ be the evaluation functional, $\delta_\gamma(h) = h(\gamma)$, $h \in B(\hat{G})$. Using the identification (4), we let $L_{\delta_\gamma} : B^n(\hat{G}) \to B^{n-1}(\hat{G})$ be the corresponding slice map. We have that $L_{\delta_\gamma}(f) \in B^{n-1}(\hat{G}_d)$ and that $L_{\delta_\gamma}(f)$ is continuous. By the induction assumption, $L_{\delta_\gamma}(f) \in B^{n-1}(\hat{G})$. Since every element of $B(\hat{G}_d)^*$ can be approximated in the weak* topology by a bounded net consisting of linear combinations of the functionals δ_γ, $\gamma \in \hat{G}$, we conclude that $L_\delta(f) \in B^{n-1}(\hat{G})$ for every $\delta \in B(\hat{G}_d)^*$. An application of [21, Theorem 2.2] shows that $f \in B(\hat{G}_d) \otimes_{eh} B^{n-1}(\hat{G})$. Repeating the above argument with a right slice map in the place of L_δ shows that $f \in B^n(\hat{G})$. $\diamondsuit$

The following lemma generalises a theorem of Eberlein [20, Theorem 1.9.1] to the multidimensional case.

Lemma 6.2. *Let $\phi \in L^\infty(\hat{G}^n)$. The following are equivalent:*
(i) $\phi \in B^n(\hat{G})$;
(ii) ϕ is continuous and there exists a constant $C > 0$ such that

$$\left|\sum c_{i_1\dots i_n}\phi(\chi_{i_1},\dots,\chi_{i_n})\right| \le C\left\|\sum c_{i_1,\dots,i_n}\chi_{i_1}\otimes\dots\otimes\chi_{i_n}\right\|_{\mathcal{V}^n(G)},$$

where $\chi_{i_k} \in \hat{G}$ and the summation is over a finite number of indices $(i_1,\dots,i_n)$.

Proof. For notational simplicity we assume $n = 2$.

(i)⇒(ii) Let $\phi \in B^2(\hat{G})$. Then by definition

$$\phi(\chi_1,\chi_2) = \tilde{\Phi}(\omega(\chi_1),\omega(\chi_2))$$

for some $\Phi \in M^2(\hat{G})$. Thus, ϕ is continuous and since $\omega(\chi_i) = \iota(\check{\chi}_i)$, where $\check{\chi}_i(x) = \overline{\chi_i(x)} = \chi_i(x^{-1})$ (see (6)), we have

$$\begin{aligned} \left|\sum c_{ij}\phi(\chi_i, \chi_j)\right| &= \left|\tilde{\Phi}(\sum c_{ij}\iota(\check{\chi}_i) \otimes \iota(\check{\chi}_j)\right| \\ &\leq \|\Phi\| \left\|\sum c_{ij}\iota(\check{\chi}_i) \otimes \iota(\check{\chi}_j)\right\|_{C_0(G)^{**}\otimes_{\rm h} C_0(G)^{**}} \\ &= \|\Phi\| \left\|\sum c_{ij}\chi_i \otimes \chi_j\right\|_{\mathcal{V}^2(G)}. \end{aligned}$$

The last equality follows from the injectivity of the Haagerup tensor product.

(ii)$\Rightarrow$(i) Assume first that G is compact. Then $\hat{G}$ is discrete. Let $T : C_0(G) \odot C_0(G) \to \mathbb{C}$ be the mapping given by $T(\sum c_{ij}\chi_i \otimes \chi_j) = \sum c_{ij}\phi(\chi_i, \chi_j)$. Then $|T(f)| \leq C\|f\|_{\mathcal{V}^2(G)} = C\|f\|_{V^2_{\rm h}(G)}$ for finite sums $f = \sum c_{ij}\chi_i \otimes \chi_j$ and therefore T can be extended to a bounded linear functional on $V^2_{\rm h}(G)$. Thus, there exists $u \in M^2(\hat{G})$ such that

$$\sum c_{ij}\phi(\chi_i, \chi_j) = \langle u, \sum c_{ij}\chi_i \otimes \chi_j\rangle.$$

In particular, $\phi(\chi_1, \chi_2) = \langle u, \chi_1 \otimes \chi_2\rangle$, that is, $\phi = \hat{u}_1 \in B^2(\hat{G})$, where $\langle u_1, \chi_i \otimes \chi_j\rangle = \langle u, \check{\chi}_i \otimes \check{\chi}_j\rangle$.

If G is not compact let $\bar{G}$ be the Bohr compactification of G. Extending each $\chi \in \hat{G}$ to a character on $\bar{G}$ we define a linear functional T on the space of all functions f on $\bar{G} \times \bar{G}$ of the form $f(x, y) = \sum c_{ij}\chi_i(x)\chi_j(y)$, $x, y \in \bar{G}$, where $\chi_i, \chi_j \in \hat{G}$, by letting, for f as above, $T(f) = \sum c_{ij}\phi(\chi_i, \chi_j)$. Let $i \in \mathbb{N}$, $g_i = \sum_k c^i_k\chi_{k,i}$ and $h_i = \sum_j d^i_j\psi_{j,i}$ be trigonometric polynomials on $\bar{G}$, where $\chi_{k,i}, \psi_{j,i} \in \hat{G}$. Then

$$\begin{aligned} \left|T\left(\sum_i g_i \otimes h_i\right)\right| &= \left|\sum_{i,k,j} c^i_k d^i_j \phi(\chi_{k,i}, \psi_{j,i})\right| \leq C \left\|\sum_{i,k,j} c^i_k d^i_j \chi_{k,i} \otimes \psi_{j,i}\right\|_{\mathcal{V}^2(G)} \\ &= C \left\|\sum_i g_i \otimes h_i\right\|_{\mathcal{V}^2(G)} = C \left\|\sum_i g_i \otimes h_i\right\|_{V^2_{\rm h}(\bar{G})}. \end{aligned}$$

The last equality follows from the injectivity of the Haagerup tensor product and the fact that $C_b(G)$ is completely isometrically embedded in $C(\bar{G})$. Thus, T can be extended to a bounded linear functional on $V^2_{\rm h}(\bar{G})$ and hence $\phi(\chi_1, \chi_2) = \langle u, \chi_1 \otimes \chi_2\rangle$ for $u \in M^2(\hat{\bar{G}}) = M^2(\hat{G}_d)$, and $\phi \in B^2(\hat{G}_d)$. Since ϕ is continuous, Proposition 6.1 implies that $\phi \in B^2(\hat{G})$. $\diamondsuit$

The following lemma is a multidimensional version of [20, Theorem 3.8.1].

Lemma 6.3. *Let* $\varphi \in L^\infty(G^n)$. *Assume* $\varphi\theta(g) \in B^n(G)$ *for every* $g \in A(G)$. *Then* $\varphi \in B^n(G)$.

Proof. We only consider the case $n = 2$; the general case can be treated in a similar way. Let $T : A(G) \to B^2(G)$ be the linear mapping defined by $T(g) = \varphi\theta(g)$. We show that T is continuous. If $g_n \to g$ in $A(G)$ and $\varphi\theta(g_n) \to \hat{u}$ in $B^2(G)$, where $u \in M^2(G)$, then

$$\hat{u}(h_1, h_2) = \lim_{n\to\infty} \varphi(h_1, h_2)g_n(h_1h_2) = \varphi(h_1, h_2)g(h_1h_2),$$

hence $\hat{u} = \varphi\theta(g)$. By the Closed Graph Theorem, T is continuous and $\|\varphi\theta(g)\|_{B^2(G)} \leq C\|g\|_{A(G)}$.

Given $h_1, \dots, h_n \in G$, $\varepsilon > 0$, there exists $f \in A(G)$, $\|f\|_{A(G)} \leq 1+\varepsilon$, such that $f(h_ih_j) = 1$, for all i, j. Let $u \in M^2(G)$ be such that $\hat{u} = \varphi\theta(f)$. Then

$$\begin{aligned}\left|\sum c_{ij}\varphi(h_i, h_j)\right| &= \left|\sum c_{ij}\varphi(h_i, h_j)f(h_ih_j)\right| = \left|\sum c_{ij}\hat{u}(h_i, h_j)\right| \\ &= \left|\tilde{u}(\sum c_{ij}\iota(\check{h}_i) \otimes \iota(\check{h}_j))\right| \\ &\leq C(1+\varepsilon)\left\|\sum c_{ij}h_i \otimes h_j\right\|_{\mathcal{V}^2(\hat{G})},\end{aligned}$$

where $\tilde{u}$ is the extention of u to a normal completely bounded linear map from $(C_0(G)^{**})^n$ to $\mathbb{C}$ and $\iota : C_b(G) \to C_0(G)^{**}$ is the canonical inclusion. Given open sets V_1, $V_2 \subset G$ with compact closures we can find $f \in A(G)$ such that $\theta(f)$ is constant on $V_1 \times V_2$. Therefore, φ is continuous on $V_1 \times V_2$, and hence φ is continuous on $G \times G$. By Lemma 6.2, $\varphi \in B^2(G)$. $\diamondsuit$

In the next corollary, we denote by M_g the operator of multiplication by the function g.

Theorem 6.4. *For every block* (k, n)*-partition* $\mathcal{P}$*, we have that* $B^n(G) = M^{cb}_{\mathcal{P}}(G) = M_{\mathcal{P}}(G)$.

Proof. Let $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) be the block $(1, n)$- (resp. (n, n)-)partition. We have that $\theta_{\mathcal{P}_2}$ is the identity map. For any block (k, n)-partition $\mathcal{P}$ we have that

$$\operatorname{ran}\theta_{\mathcal{P}_1} \subseteq \operatorname{ran}\theta_{\mathcal{P}} \subseteq \operatorname{ran}\theta_{\mathcal{P}_2} = A^n(G).$$

Thus,

$$M_{\mathcal{P}_2}A(G) \subseteq M_{\mathcal{P}}A(G) \subseteq M_{\mathcal{P}_1}A(G),$$

and similarly for the completely bounded multipliers. By Theorem 5.1, $B^n(G) \subseteq M_{\mathcal{P}_2}A(G)$. By Lemma 6.3, $M_{\mathcal{P}_1}A(G) \subset B^n(G)$ and hence $B^n(G) = M_{\mathcal{P}}(G)$.

The fact that $B^n(G) = M^{cb}_{\mathcal{P}}A(G)$ follows in the same way, using Proposition 4.2. $\diamondsuit$

Corollary 6.5. *Let $\Psi : A(G) \to A^n(G)$ be a bounded linear map such that $\Psi M_\chi = M_{\theta(\chi)}\Psi$ for any $\chi \in \hat{G}$. Then $\Psi(f) = \varphi\theta(f)$, $f \in A(G)$, for some $\varphi \in B^n(G)$.*

Proof. It follows from the proof of Theorem 5.12 that $\Psi(f) = \varphi\theta(f)$ for some bounded function φ on G. Thus $\varphi \in M_nA(G)$. The statement now follows from Theorem 6.4. $\diamondsuit$

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