

THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

# On properties of Brownian interlacements and Brownian excursions

Olof Elias



**CHALMERS**  
UNIVERSITY OF TECHNOLOGY



UNIVERSITY OF GOTHEN

Division of Analysis and Probability  
Department of Mathematical Sciences  
University of Gothenburg and Chalmers University of Technology  
Göteborg, Sweden 2018

On properties of Brownian interlacements and Brownian excursions  
Olof Elias  
Göteborg 2018

© Olof Elias, 2018

Division of Analysis and Probability  
Department of Mathematical Sciences  
University of Gothenburg and Chalmers University of Technology  
SE-412 96 Göteborg  
Sweden  
Telephone +46 31 772 53 16

Typeset with  $\text{\LaTeX}$   
Printed by BrandFactory AB, Sweden 2017

# On properties of Brownian interlacements and Brownian excursions

Olof Elias

Division of Analysis and Probability  
Department of Mathematical Sciences  
University of Gothenburg and Chalmers University of Technology

## Abstract

This thesis deals with two relatively new continuum percolation models, the Brownian interlacements and the Brownian excursions. Locally, these models look like a Poisson distributed number of transient Brownian motions. Due to the nature of these models classical techniques from percolation theory are often not applicable. More precisely, both models exhibit infinite range dependence.

In the appended paper we study visibility in the vacant set of the Brownian interlacements, that is the existence of geodesic line-segments of some fixed length  $r$  contained in the vacant set, emanating from some fixed point. We give upper and lower bounds on the probability of the existence of such line segments in terms of  $r$ . We also consider the Brownian excursions model in the unit disk and show that it undergoes a non-trivial phase transition concerning visibility to infinity (in the hyperbolic metric).

In Chapter 3 of the thesis we show that in the occupied region of Brownian interlacements the probability of having visibility in a fix direction of length  $r$  decays exponentially for intensities low enough.

Finally, in Chapter 4 we prove that the vacant set of the Brownian excursions model in the unit disk has a non-trivial phase transition concerning percolation and that the distribution of the Brownian excursions can be described in a similar way to that of the Brownian interlacements.

**Keywords:** continuum percolation, brownian interlacements, brownian excursions, visibility



## List of publications

- I. **Elias, O.**, Tykesson J. (2017). Visibility in the vacant set of the Brownian interlacements and the Brownian excursion process. *Submitted*. Preprint available at arXiv:1709.09052



# Acknowledgements

It goes without saying that this thesis would not have been possible without the help, support and encouragement from my supervisor **Johan Tykesson**. Thank you for always having time to discuss anything remotely related to percolation and your constant reminder that the devil is in the details. I want to thank my co-supervisor **Johan Jonasson** for all the small talks we've had. I would also like to thank the Probability group, as a whole, for the wonderful workshop at Aspenäs Herrgård.

Thanks to Malin, Fanny, Jonatan, Viktor, Anna J, Mariana, and everyone else that are or have been a part of Lämmeltåget. Thank you Ivar, Claes, Sandra, Henrik, Linnea, Marco, Edvin, Anders M and H, and all the other wonderful people at the institution. Thanks to Marija and the Biostat group for the summer project back in 2012 which made me consider an academic career in the first place.

Finally, I want to thank my family and friends, especially my niece and nephews Annika, Ture, Martin and Karl, for all the pleasant distractions.

Thank you Filippa, my love, you make everything feel worthwhile.



# Contents

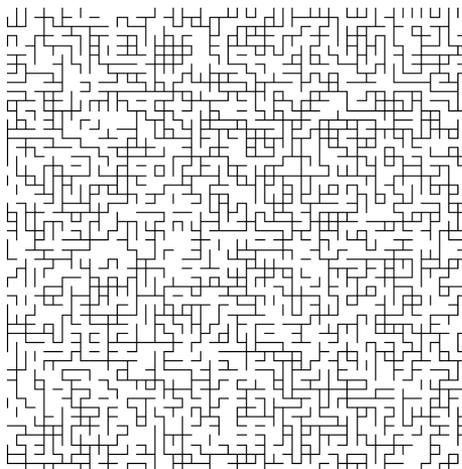
<b>Abstract</b>	<b>iii</b>
<b>List of publications</b>	<b>v</b>
<b>Acknowledgements</b>	<b>vii</b>
<b>Contents</b>	<b>ix</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Percolation theory</b>	<b>5</b>
2.1 Notation . . . . .	5
2.2 Stochastic geometry and Poisson point processes . . . . .	5
2.3 The random interlacements model . . . . .	6
2.4 The Brownian interlacements in Euclidean space . . . . .	9
2.5 Brownian motion and the hyperbolic plane . . . . .	13
2.6 Brownian excursions in the unit disc . . . . .	15
<b>3 Visibility in the Brownian interlacements</b>	<b>19</b>
3.1 Long range dependence and visibility in the Brownian interlacements	19
<b>4 Percolation in the vacant set of the Brownian excursions</b>	<b>25</b>
4.1 Brownian excursions and Brownian interlacements . . . . .	25
4.2 Percolation in the vacant set . . . . .	28

<b>5 Summary of papers</b>	<b>33</b>
5.1 Paper I . . . . .	33
<b>Bibliography</b>	<b>35</b>
<b>Paper I</b>	

# 1 Introduction

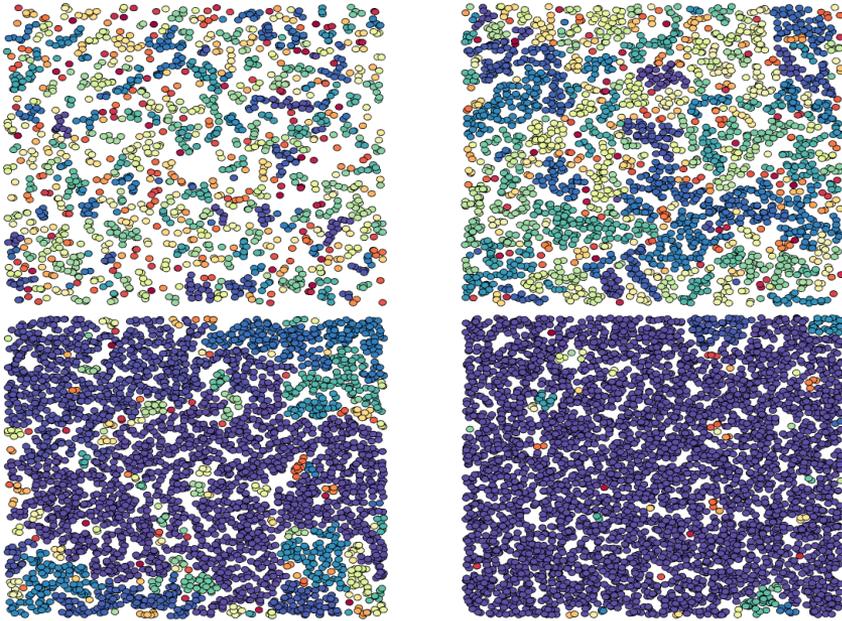
Percolation theory, in its essence, is about studying random systems that undergo macroscopic changes when altering its microscopic behavior. Usually a random system is described by some “local rule” that determines the microscopic behavior of the system. Classical examples of phenomena that can be analyzed using percolation theory are ferromagnetism, crystal formation and communication systems, see for instance [1], [2] and [3].

The name *percolation theory* appeared first in [4] where the now classical bond percolation model was introduced. An intuitive description of this model on the lattice  $\mathbb{Z}^2$  is as follows: for each edge flip a, not necessarily fair, coin and remove the edge if the coin shows tails and keep the edge if the coin shows heads. One is then interested in studying various geometric aspects of the random graph created in this fashion. For example, one is often interested in the existence of an infinite connected component of this random graph. When this occurs, we say that the model percolates. Despite the fact that this model is relatively easy to formulate, understanding it is different question.



**Figure 1.1:** Bond percolation on  $\mathbb{Z}^2$  with  $p = 0.51$ , obtained from [5]

There are numerous variations of this model, such as the Poisson Boolean continuum percolation model and random-cluster model. For a more in-depth discussion of these models we refer to [2], [3], [6] and [7]. In Figure 1.2 different simulations of the Poisson Boolean model are given, for four different intensities. Moreover, note that the largest cluster of the Poisson Boolean model varies quite dramatically for different values of the intensity parameter.



**Figure 1.2:** The Poisson Boolean model. Different colors represents different clusters.

Several of these models share one property, the finite range dependence. This means that if we look at two different regions that lie sufficiently far apart, whatever happens in these regions will happen independently of each other. On the other hand, the other models that *do* have correlations usually exhibit exponential decay of the correlation functions.

In recent times a lot of attention has been given to the random interacements model, a site-percolation model on  $\mathbb{Z}^d$ ,  $d \geq 3$ , that does not have this property. In fact the decay of correlations is of polynomial order. This complicates the application of standard arguments and even proving that this model undergoes a non-trivial phase transition concerning percolation in its vacant set is complicated, see [8] and [9]. The proofs often rely on so called multiscale renormalization techniques to circumvent the infinite-range dependence of this model.

A continuum analogue of the random interacements, called the Brownian interacements, was introduced in [10] as a means to study the scaling limits of the oc-

cupation time measure of continuous time random interlacements. The properties of the Brownian interlacements, in comparison with the random interlacements, are to a large extent unknown, but some of the arguments are transferable. For instance, in [11] the existence of a non-trivial phase transition concerning percolation in the vacant set is proven and connectivity properties of the occupied set are studied.

The remainder of this thesis consists of three chapters and one appended paper. Chapter 2 introduces the models studied in this thesis, namely the Brownian interlacements and the Brownian excursions process and their properties. We also briefly discuss the random interlacements model and state some of the results concerning this process.

Chapters 3 and 4 consist of new results that can be considered as by-products of the article [12]. Chapter 3 consists of results regarding visibility and infinite range dependence *in* the Brownian interlacements set. We give bounds for the 2-point correlation function for the interlacement set, which resembles known results for the random interlacements. Moreover, we show that for intensities low enough the visibility in a fixed direction decays exponentially for the interlaced set, something that is true for the vacant set in general as seen in [12].

Chapter 4 consists of results regarding the Brownian excursions in the unit disk. This process is invariant under the isometries of the Poincaré disk model of the hyperbolic plane. We show that the local structure can be described in a similar way as that of the local structure of the Brownian interlacements in Euclidean space. We also show that the vacant set undergoes a phase transition regarding percolation using a connection with the canonical Poisson line process on the hyperbolic plane.

The appended paper, [12], is the main part of this thesis and a summary of the paper is given in Chapter 5.



# 2 Percolation theory

This thesis deals with different models of continuum percolation and several areas of probability theory come into play. The three major actors are random closed sets, Poisson processes and Brownian motion and its potential theory. Since the Brownian interlacements can be considered to be the continuum counterpart of the random interlacements, we discuss this model as well.

## 2.1 Notation

We need to introduce some notation that we will use throughout the thesis. For a topological space  $X$  we write  $K \Subset X$  to indicate that  $K$  is compact. For functions  $f, g : X \rightarrow \mathbb{R}_+$  we write  $f \asymp g$  to indicate that there exists constants  $c, c'$  such that  $cg \leq f \leq c'g$ . If  $X$  is a metric space with metric  $d$  we write

$$A^t := \{x \in X : \text{dist}(x, A) \leq t\}$$

for the closed  $t$ -neighbourhood of  $A$ . Finally a convention about constants. We shall let constants depend on the dimension of the space, if it depends on any other quantity  $x$  we shall emphasize this by writing  $c(x)$ .

## 2.2 Stochastic geometry and Poisson point processes

The Brownian interlacements set is an example of a random closed set in  $\mathbb{R}^d$  for  $d \geq 3$ . In general, we define a random closed set in a metric space  $(S, d)$  as a probability measure on the Fell topology of closed sets, see [13]. More precisely, we let  $\Sigma$  be the collection of closed sets in  $S$  with respect to the metric  $d$  and let

$$\mathbb{F} := \sigma(F \in \Sigma : F \cap K = \emptyset, K \text{ compact}).$$

We then define a random closed set as a probability measure  $P$  on  $(\Sigma, \mathbb{F})$ .

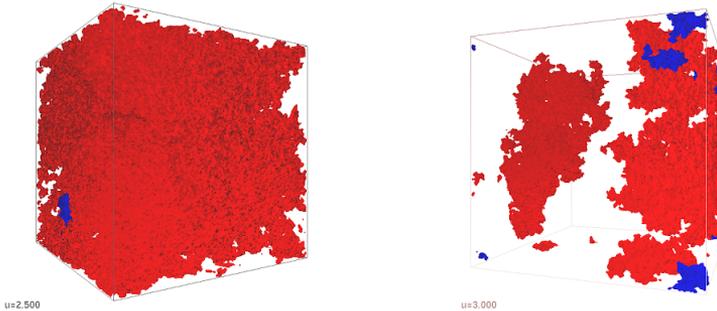
Now suppose  $(S, d)$  is a complete separable metric space. Then we can define a random measure as a probability measure on the space of measures on  $S$ , denoted by  $\mathcal{P}(S)$ , endowed with the topology of weak convergence, see [14].

Moreover, let  $\mu$  be a measure on  $(S, \mathcal{S})$ , where  $\mathcal{S}$  is the Borel sigma-algebra. We then define the Poisson point process,  $X$ , on  $(S, d)$  with intensity measure  $\mu$ , to be the locally finite random measure satisfying

- For any collection of disjoint measurable sets  $\{A_i\}_{i=1}^n$  the random variables  $X(A_i)$  are all mutually independent.
- $P(X(A) = n) = \frac{\mu(A)^n}{n!} e^{-\mu(A)}$ ,  $n \in \{0, 1, 2, \dots\}$ .

## 2.3 The random interlacements model

The random interlacements model was introduced by A.S Sznitman in [8] as a novel percolation model exhibiting infinite range dependence. Informally, this model consists of a Poisson process on the space of doubly-infinite transient nearest-neighbour paths modulo time-shifts, which looks like two-sided simple random walks. The development of the model was motivated by the study of disconnection times of random walks on the discrete torus  $\mathbb{Z}^d \setminus NZ^d$  in [15].



**Figure 2.1:** The vacant set of the Random interlacements in  $\mathbb{Z}^3$  for different values of  $\alpha$ . The largest cluster is colored red. Used with permission of the author. Obtained from [16]

For completeness sake we shall give the definition of the random interlacements to highlight similarities as well as differences between this model and the Brownian interlacements. Let  $\mathbb{Z}^d$  be the integer lattice and let  $\|\cdot\|_1$  be the  $L^1$  norm on  $\mathbb{Z}^d$ , that is

$$\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i|, \quad x, y \in \mathbb{Z}^d.$$

Let

$$E = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \|x - y\|_1 = 1\}$$

be the set of edges and consider the graph  $G = (V, E)$  where  $V = \mathbb{Z}^d$ . On this graph we define

$$\Pi_+ = \left\{ w : \mathbb{N} \rightarrow \mathbb{Z}^d : \|w(n) - w(n+1)\|_1 = 1, \forall n \in \mathbb{N}, \lim_{n \rightarrow \infty} \|w(n)\|_1 = \infty \right\}$$

and

$$\Pi = \left\{ w : \mathbb{Z} \rightarrow \mathbb{Z}^d : \|w(n) - w(n+1)\|_1 = 1, \forall n \in \mathbb{N}, \lim_{|n| \rightarrow \infty} \|w(n)\|_1 = \infty \right\}$$

to be the set of all infinite and doubly-infinite transient nearest-neighbour paths on the graph  $G$ . Let  $X_n$  be the canonical coordinates on  $\Pi$  or  $\Pi_+$ , that is  $X_n(w) = w(n)$  for  $w \in \Pi$  or  $\Pi_+$ . Let  $\sigma(\Pi)$  and  $\sigma(\Pi_+)$  denote the sigma-algebra generated by the canonical coordinates  $(X_n)_n$ . Moreover, let  $\theta_n$  be the shifts on  $\mathbb{N}$  or  $\mathbb{Z}$ . For  $K \Subset \mathbb{Z}^d$  define

$$\Pi_K = \left\{ w \in \Pi : \exists n \in \mathbb{Z}^d \text{ such that } w(n) \in K \right\}$$

to be the set of all doubly infinite trajectories that hit  $K$ .

For  $w, w' \in \Pi$  introduce the equivalence relation

$$w \sim w' \Leftrightarrow \exists k \in \mathbb{Z} : w = w' \circ \theta_k.$$

Let

$$\Pi^* = \Pi / \sim$$

be the quotient space and let

$$p : \Pi \rightarrow \Pi^*$$

denote the canonical projection. Moreover, let  $\sigma(\Pi^*)$  be the smallest sigma-algebra such that  $p$  is measurable, that is

$$\sigma(\Pi^*) = \left\{ A \subset \Pi^* : p^{-1}(A) \in \sigma(\Pi) \right\},$$

and let  $\Pi_K^* := p(\Pi_K)$ . Now, for  $U \subset \mathbb{Z}^d$  and  $w \in \Pi_+$  let

$$H_U(w) = \inf \{ n \geq 0 : X_n(w) \in U \}$$

be the entrance time and let

$$\tilde{H}_U = \inf \{ n \geq 1 : X_n(w) \in U \}$$

denote the hitting time. Let  $S_x$  be the symmetric simple random walk measure supported on paths  $w \in \Pi_+$  such that  $w(0) = x$  and we let  $S_x^K(\cdot) = S_x(\cdot | \tilde{H}_K = \infty)$  be the excursion measure. Moreover, for a finite positive measure  $\rho$  define

$$S_\rho = \sum_{x \in \mathbb{Z}^d} \rho(x) S_x.$$

Now we define the equilibrium measure on a finite set  $K \Subset \mathbb{Z}^d$  as

$$e_K^{\mathbb{Z}}(x) = \begin{cases} S_x(\tilde{H}_K = \infty), & x \in K, \\ 0 & x \notin K. \end{cases} \quad (2.1)$$

It should be clear from the definition that  $e_K^{\mathbb{Z}}(x)$  is a finite measure supported on the boundary vertices of  $K$ . For clarity as well as convenience, we write  $\mathbb{Z}$  instead of  $\mathbb{Z}^d$  to emphasize that all quantities given here are related to the random inter-

placements model on  $\mathbb{Z}^d$  rather than the Brownian interacements. We then define the capacity of a finite set  $K$  as

$$\text{cap}_{\mathbb{Z}}(K) = e_K^{\mathbb{Z}}(K).$$

Now, for  $K \Subset \mathbb{Z}^d$ ,  $x \in \mathbb{Z}^d$  and  $A, B \in \sigma(\Pi_+)$  we define the following finite measures

$$Q_K^{\mathbb{Z}}((X_n)_{n \leq 0} \in A, X_0 = x, (X_n)_{n \geq 0} \in B) := S_x^K(A) e_K^{\mathbb{Z}}(x) S_x(B).$$

Then Theorem 1.1 on p.2049 in [8] states the following.

**Theorem 2.3.1.** *There exists a unique  $\sigma$ -finite measure  $\nu_{\mathbb{Z}}$  on  $(\Pi^*, \sigma(\Pi^*))$  such that*

$$\nu_{\mathbb{Z}}(\cdot \cap \Pi_K^*) = p \circ Q_K^{\mathbb{Z}}. \quad (2.2)$$

This fact allows us to construct a Poisson process on the space  $\Pi^* \times \mathbb{R}_+$  as follows. Let

$$\Omega_{\mathbb{Z}} := \left\{ \omega = \sum_{i \geq 0} \delta_{(w_i^*, \alpha_i)} : \omega(\Pi_K^* \times [0, \alpha]) < \infty, \forall K \Subset \mathbb{Z}^d, \alpha \geq 0 \right\},$$

and let  $\mathcal{M}_{\mathbb{Z}}$  be the  $\sigma$ -algebra generated by the evaluation maps

$$\omega \mapsto \omega(B), \quad B \in \sigma(\Pi^*) \times \mathcal{B}(\mathbb{R}_+).$$

Finally, let  $\mathbb{P}_{\mathbb{Z}}$  be the law of the Poisson process on  $\Omega_{\mathbb{Z}}$  with intensity measure  $\nu_{\mathbb{Z}}$ . Now we can define the interlacement set at level  $\alpha$  as the subset  $\mathbb{Z}^d$  given by

$$\text{Rl}_{\alpha} = \bigcup_{\alpha_i \leq \alpha} \text{range}(w_i^*), \quad \omega = \sum_{i \geq 0} \delta_{(w_i^*, \alpha_i)}, \quad (2.3)$$

and the vacant set is defined as

$$\mathcal{V}_{\alpha, \mathbb{Z}} := \mathbb{Z}^d \setminus \text{Rl}_{\alpha}.$$

The law of  $\mathcal{V}_{\alpha, \mathbb{Z}}$ , which we denote by  $Q_{\alpha}^{\mathbb{Z}}$ , is given by the equality

$$Q_{\alpha}^{\mathbb{Z}}\left(\left\{ \omega \in \{0, 1\}^{\mathbb{Z}^d} : \omega(z) = 1, \forall z \in K \right\}\right) = \mathbb{P}_{\mathbb{Z}}(\text{Rl}_{\alpha} \cap K = \emptyset) = e^{-\alpha \text{cap}_{\mathbb{Z}}(K)} \quad (2.4)$$

for  $K$  finite, see Proposition 1.5 on page 2055 in [8]. To give an intuition about  $\text{Rl}_{\alpha}$  note that since

$$Q_K^{\mathbb{Z}}((X_t)_{t \geq 0} \in \cdot) = S_{e_K}(\cdot)$$

for  $K \Subset \mathbb{Z}^d$  the random set  $\text{Rl}_{\alpha} \cap K$  satisfies the following distributional equality

$$\text{Rl}_{\alpha} \cap K \stackrel{d}{=} \bigcup_{i=1}^{N_K} [w_i] \cap K \quad (2.5)$$

where  $N_K \sim \text{Po}(\alpha \text{cap}_{\mathbb{Z}}(K))$  and  $[w_i]$  is the trace of i.i.d simple random walks with initial distribution  $e_K(\cdot)/\text{cap}_{\mathbb{Z}}(K)$ .

Define the percolation function

$$\eta_{\mathbb{Z}}(\alpha) = \mathbb{P}_{\mathbb{Z}}(0 \text{ belongs to an infinite connected component of } \mathcal{V}_{\alpha, \mathbb{Z}}), \quad (2.6)$$

and let

$$\alpha_c^{\mathbb{Z}} = \inf \{ \alpha \geq 0 : \eta_{\mathbb{Z}}(\alpha) = 0 \}$$

be the critical parameter. From [8] and [9] we know the following.

**Theorem 2.3.2.** *The critical parameter is non-trivial:*

$$\alpha_c^{\mathbb{Z}} \in (0, \infty). \quad (2.7)$$

A shorter proof of this fact is available in [17]. Moreover, at the supercritical regime,  $\alpha > \alpha_c$ , we know that there is a unique infinite connected component in the vacant set.

**Theorem 2.3.3.** *For  $\alpha \geq 0$  there exists at most one infinite component of  $\mathcal{V}_{\alpha, \mathbb{Z}}$ .*

This result is found in [18]. Additionally, in [19] the model was shown to be well-defined on general transient graphs, and the law of the vacant set  $Q_{\alpha}^{\mathbb{Z}}$  satisfies the FKG-inequality:

**Theorem 2.3.4.** *Let  $f, g : \{0, 1\}^{\mathbb{Z}^d} \rightarrow [0, 1]$  be increasing functions, then*

$$\int f g dQ_{\alpha}^{\mathbb{Z}} \geq \int f dQ_{\alpha}^{\mathbb{Z}} \int g dQ_{\alpha}^{\mathbb{Z}}.$$

Finally, connectivity properties of the interlacement set were established in [20] and [21]. The main result there was:

**Theorem 2.3.5.** *Any two vertices in  $R|_{\alpha}$  are connected by the union of the traces of at most  $\lceil d/2 \rceil$  trajectories in the Poisson process  $\omega_1 \{ \alpha_i \leq \alpha \}$ .*

## 2.4 The Brownian interlacements in Euclidean space

We now move on to the Brownian interlacements, the continuum counterpart of the random interlacements. Similarly to the random interlacements, this model is, informally, a Poisson process on the space of doubly-infinite Brownian paths.

The construction presented here is essentially taken from [12]. Let  $C = C(\mathbb{R}; \mathbb{R}^d)$  denote the continuous functions from  $\mathbb{R}$  to  $\mathbb{R}^d$  and let  $C_+ = C(\mathbb{R}_+; \mathbb{R}^d)$  denote the continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}^d$ . Let  $\|\cdot\|_2$  be the Euclidean norm on  $\mathbb{R}^d$  and define

$$W = \left\{ x \in C : \lim_{|t| \rightarrow \infty} \|x(t)\|_2 = \infty \right\} \text{ and } W_+ = \left\{ x \in C_+ : \lim_{t \rightarrow \infty} \|x(t)\|_2 = \infty \right\}.$$

On  $W$  we let  $X_t$ ,  $t \in \mathbb{R}$ , denote the canonical process, i.e.  $X_t(w) = w(t)$  for  $w \in C$ , and let  $\mathcal{W}$  denote the  $\sigma$ -algebra generated by the cylinder sets of the canonical

processes. Moreover we let  $\theta_h, h \in \mathbb{R}$  denote the shift operators acting on  $\mathbb{R}$ , that is  $\theta_h : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto y + h$ . We extend this notion to act on  $C$  by composition as

$$\theta_h : C \rightarrow C, f \mapsto f \circ \theta_h.$$

Similarly, on  $W_+$ , we define the canonical process  $X_t, t \geq 0$ , the shifts  $\theta_h, h \geq 0$ , and the sigma algebra  $\mathcal{W}_+$  generated by the canonical processes. We define the following random times corresponding to the canonical processes. For  $F \subset \mathbb{R}^d$  closed and  $w \in W_+$ , the **entrance time** is defined as

$$H_F(w) = \inf\{t \geq 0 : X_t(w) \in F\}$$

and the **hitting time** is defined as

$$\tilde{H}_F(w) = \inf\{t > 0 : X_t(w) \in F\}.$$

For  $K \Subset \mathbb{R}^d$  the **time of last visit** to  $K$  for  $w \in W_+$  is defined as

$$L_K(w) = \sup\{t > 0 : X_t(w) \in K\}.$$

The entrance time for  $w \in W$  is defined similarly, but  $t > 0$  is replaced by  $t \in \mathbb{R}$ . On  $W$ , we introduce the equivalence relation  $w \sim w' \Leftrightarrow \exists h \in \mathbb{R} : \theta_h w = w'$  and we denote the quotient space by  $W^* = W / \sim$  and let

$$\pi : W \rightarrow W^*, w \mapsto w^*,$$

denote the canonical projection. Moreover, we let  $\mathcal{W}^*$  denote the smallest  $\sigma$ -algebra such that  $\pi$  is a measurable function, i.e.  $\mathcal{W}^* = \{\pi^{-1}(A) : A \in \mathcal{W}\}$ . We denote by  $W_K \subset W$  all trajectories which enter  $K$ , and  $W_K^*$  the associated projection. We let  $P_x$  be the Wiener measure on  $C$  with the canonical process starting at  $x$ , and we define  $P_x^B(\cdot) = P_x(\cdot | H_B = \infty)$  to be the probability measure conditioned on the event that the Brownian motion never hits  $B$ . For a finite measure  $\lambda$  on  $\mathbb{R}^d$  we define

$$P_\lambda = \int P_x \lambda(dx).$$

The transition density for the Brownian motion on  $\mathbb{R}^d$  is given by

$$p(t, x, y) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x - y\|_2^2}{2t}\right) \quad (2.8)$$

and the Green's function is given by

$$G(x, y) = G(x - y) := \int_0^\infty p(t, x, y) dt = c_d / \|x - y\|_2^{d-2},$$

where  $c_d = \Gamma(d/2 - 1) / 2\pi^{d/2}$ , see Theorem 3.33 p.80 in [22].

Following [10] we introduce the following potential theoretic framework. For  $K \Subset \mathbb{R}^d$  let  $\mathcal{P}(K)$  be the space of probability measures supported on  $K$  and introduce the energy functional

$$E_K(\lambda) = \int_{K \times K} G(x, y) \lambda(dx) \lambda(dy), \lambda \in \mathcal{P}(K). \quad (2.9)$$

The Newtonian capacity of  $K \Subset \mathbb{R}^d$  is defined as

$$\text{cap}(K) := \left( \inf_{\lambda \in \mathcal{P}(K)} \{E_K(\lambda)\} \right)^{-1}, \quad (2.10)$$

see for instance [23], [24] or [22]. It is the case that

$$\text{the capacity is a strongly sub-additive and monotone set-function.} \quad (2.11)$$

Let  $e_K(dy)$  be the equilibrium measure, which is the finite measure that is uniquely determined by the last exit formula, see Theorem 8.8 in [22],

$$P_x(X(L_K) \in A) = \int_A G(x, y) e_K(dy), \quad (2.12)$$

and let  $\tilde{e}_K$  be the normalized equilibrium measure. By Theorem 8.27 on p. 240 in [22] we have that  $\tilde{e}_K$  is the unique minimizer of (2.9) and

$$\text{cap}(K) = e_K(K). \quad (2.13)$$

Moreover the support satisfies  $\text{supp } e_K = \partial K$ .

If  $B$  is a closed ball, we define the measure  $Q_B$  on  $W_B^0 := \{w \in W : H_B(w) = 0\}$  as follows:

$$Q_B[(X_{-t})_{t \geq 0} \subset A', X_0 \in dy, (X_t)_{t \geq 0} \subset A] := P_y^B(A') P_y(A) e_B(dy), \quad (2.14)$$

where  $A, A' \in \mathcal{W}_+$ . If  $K$  is compact, then  $Q_K$  is defined as

$$Q_K = \theta_{H_K} \circ (1\{H_K < \infty\} Q_B), \text{ for any closed ball } B \supseteq K.$$

As pointed out in [10] this definition is independent of the choice of  $B \supseteq K$  and coincides with (2.14) when  $K$  is a closed ball. We point out that Equation 2.21 of [10] says that

$$Q_K[(X_t)_{t \geq 0} \in \cdot] = P_{e_K}(\cdot). \quad (2.15)$$

From [10] we have the following theorem, which is Theorem 2.2 on p.564.

**Theorem 2.4.1.** *There exists a unique  $\sigma$ -finite measure  $\nu$  on  $(W^*, \mathcal{W}^*)$  such that for all  $K$  compact,*

$$\nu(\cdot \cap W_K^*) = \pi \circ Q_K(\cdot) \quad (2.16)$$

Now we introduce the space of point measures or configurations, where  $\delta$  is the usual Dirac measure:

$$\Omega = \left\{ \omega = \sum_{i \geq 0} \delta(w_i^*, \alpha_i) : (w_i^*, \alpha_i) \in W^* \times [0, \infty), \omega(W_K^* \times [0, \alpha]) < \infty, \forall K \Subset \mathbb{R}^d, \alpha \geq 0 \right\},$$

and we endow  $\Omega$  with the  $\sigma$ -algebra  $\mathcal{M}$  generated by the evaluation maps

$$\omega \mapsto \omega(B), B \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+).$$

Furthermore, we let  $\mathbb{P}$  denote the law of the Poisson point process of  $W^* \times \mathbb{R}_+$

with intensity measure  $\nu \otimes d\alpha$ . The Brownian interlacement is then defined as the random closed set

$$\mathbf{Bl}_\alpha^\rho(\omega) := \bigcup_{\alpha_i \leq \alpha} \bigcup_{s \in \mathbb{R}} B(w_i(s), \rho), \quad (2.17)$$

where  $\omega = \sum_{i \geq 0} \delta(w_i^*, \alpha_i) \in \Omega$  and  $\pi(w_i) = w_i^*$ . We then let  $\mathcal{V}_{\alpha, \rho} = \mathbb{R}^d \setminus \mathbf{Bl}_\alpha^\rho$  denote the vacant set.

The law of  $\mathbf{Bl}_\alpha^\rho$  is characterized as follows. Let  $\Sigma$  denote the family of all closed sets of  $\mathbb{R}^d$  and let  $\mathbb{F} := \sigma(F \in \Sigma : F \cap K = \emptyset, K \text{ compact})$ . The law of the interlacement set,  $Q_\alpha^\rho$ , is a probability measure on  $(\Sigma, \mathbb{F})$  given by the following identity:

$$Q_\alpha^\rho(\{F \in \Sigma : F \cap K = \emptyset\}) = \mathbb{P}(\mathbf{Bl}_\alpha^\rho \cap K = \emptyset) = e^{-\alpha \text{cap}(K^\rho)}. \quad (2.18)$$

In [10] it is shown that  $\mathbb{P}$  is invariant under translations, time-inversions and linear isometries, see Proposition 2.4 on p.569.

*Remark 2.1.* To get a better intuition of how this model works it might be good to think of the local structure of the random set  $\mathbf{Bl}_\alpha^\rho$ . This can be done in the following way, which uses (2.15). Let  $K \subset \mathbb{R}^d$  be a compact set. Let  $N_K \sim \text{Poisson}(\alpha \text{cap}(K))$ . Conditioned on  $N_K$ , let  $(y_i)_{i=1}^{N_K}$  be i.i.d. with distribution  $\tilde{\epsilon}_K$ . Conditioned on  $N_K$  and  $(y_i)_{i=1}^{N_K}$  let  $((B_i(t))_{t \geq 0})_{i=1}^{N_K}$  be a collection of independent Brownian motions in  $\mathbb{R}^d$  with  $B_i(0) = y_i$  for  $i = 1, \dots, N_K$ . We have the following distributional equality:

$$K \cap \mathbf{Bl}_\alpha^\rho \stackrel{d}{=} \left( \bigcup_{i=1}^{N_K} [B_i]^\rho \right) \cap K, \quad (2.19)$$

where  $[B_i]$  stands for the trace of  $B_i$ .

We end this section with some of the known results for the Brownian interlacements model. The results are similar to the corresponding results for the random interlacements model. Let

$$\eta(\alpha) = \mathbb{P}(0 \text{ belongs to an infinite connected component of } \mathcal{V}_\alpha)$$

be the percolation function and define the critical parameter for percolation in the Brownian interlacements:

$$\alpha_c = \inf\{\alpha \geq 0 : \eta(\alpha) = 0\}.$$

In [11] we have the following result:

**Theorem 2.4.2.** *The critical parameter is non trivial :*

$$\alpha_c \in (0, \infty). \quad (2.20)$$

Additionally, in the same article we also have the following result on the connectivity properties for the Brownian interlacements.

**Theorem 2.4.3.** *Any two points in  $\mathbf{Bl}_\alpha^\rho$  are connected by the union of the traces of at most  $\lceil d/2 \rceil$  trajectories in the Poisson process  $\omega 1_{\{\alpha_i \leq \alpha\}}$ , for any  $\alpha, \rho > 0$ .*

From Lemma 2.1 in [25] we also know that  $\mathbb{P}$  satisfies the FKG-inequality, though it is not known whether the trace of the Poisson process satisfies the FKG-inequality. To be clear, we do not know whether  $Q_\alpha^\rho$  satisfy the FKG-inequality.

## 2.5 Brownian motion and the hyperbolic plane

In Chapter 4 we consider the Brownian excursions model in the disk. This model is invariant under the isometries of the Poincaré disk model of the hyperbolic plane. Therefore we will give a background on the hyperbolic plane and the Brownian motion in the hyperbolic plane.

The hyperbolic plane is essentially *the* 2-dimensional manifold of constant negative curvature according to the Uniformization theorem of Riemannian surfaces. We will consider the Poincaré disk model of the hyperbolic plane which is defined by letting  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disk in the complex plane equipped with the hyperbolic metric

$$ds^2 := \frac{4}{(1 - |z|^2)^2} (dx^2 + dy^2),$$

so that the distance between  $u, v \in \mathbb{D}$  is given by

$$\rho(u, v) = 2 \tanh^{-1} \left| \frac{u - v}{1 - \bar{u}v} \right|.$$

The corresponding volume measure is given by

$$d\mu = \frac{4}{(1 - |z|^2)^2} dx dy.$$

*Remark 2.2.* There are several other models of the hyperbolic plane. The perhaps most common model is the Poincaré half-plane model which is defined as the upper half of the complex plane,

$$H = \{z \in \mathbb{C} : \text{Im } z > 0\},$$

equipped with the metric

$$ds^2 := \frac{dx^2 + dy^2}{y^2}.$$

These are seen to be isometric by applying either of the Möbius transformations

$$\begin{aligned} \phi : H &\rightarrow \mathbb{D}, z \mapsto \frac{i-z}{iz-1}, \\ \psi : \mathbb{D} &\rightarrow H, z \mapsto \frac{1+z}{1+iz}. \end{aligned} \tag{2.21}$$

Finally the isometry group of  $(\mathbb{D}, \rho)$  is given by the family of functions

$$T_{a,\lambda}(z) = \lambda \frac{z-a}{\bar{a}z-1}, \quad |\lambda| = 1, |a| < 1, \tag{2.22}$$

which incidentally is the group of conformal automorphisms on  $\mathbb{D}$ . We refer to  $\mathbb{D}$  equipped with  $\rho$  as the Poincaré disk model of the 2-dimensional hyperbolic space  $\mathbb{H}^2$ . For different models and additional facts regarding hyperbolic geometry we refer to [26].

There are several ways of constructing Brownian motion on  $\mathbb{H}^2$  and we shall give three different constructions. The most direct way might be to define it as the strong Markov process with transition density given as the smallest solution to the PDE:

$$\begin{cases} \partial_t p(t, x, y) - \frac{1}{2} \Delta_{\mathbb{H}^2} p(t, x, y) = 0, \\ \lim_{t \downarrow 0} p(t, \cdot, y) = \delta_y, \end{cases} \quad (2.23)$$

where

$$\Delta_{\mathbb{H}^2} = \frac{(1 - |z|^2)^2}{4} (\partial_x^2 + \partial_y^2). \quad (2.24)$$

The solution to (2.23) as well as the Green's function is given by:

$$p(t, x, y) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_{\rho(x, y)}^{\infty} \frac{se^{-s^2/4t}}{\sqrt{\cosh(s) - \cosh(\rho(x, y))}} ds$$

$$G_{\mathbb{H}^2}(z, w) := \frac{1}{2} \int_0^{\infty} p(t, x, y) dt = \frac{1}{2\pi} \log \frac{|1 - z\bar{w}|}{|z - w|} \quad (2.25)$$

$$= -\frac{1}{2\pi} \log(\tanh(\rho(z, w)/2)). \quad (2.26)$$

The first derivation of the heat kernel in  $\mathbb{H}^2$  is credited to McKean, [27]. We will drop the subscript  $\mathbb{H}^2$  when it is clear from context. Also, note that the hyperbolic Green's function is the same as the Green's function of a Brownian motion started in  $\mathbb{D}$  stopped upon hitting  $\partial\mathbb{D}$ .

A different way of defining the hyperbolic Brownian motion on  $\mathbb{D}$  is to define it as the conformally invariant diffusion on  $\mathbb{D}$  given by

$$dX(t) = \frac{1}{2}(1 - |X(t)|^2)dZ(t)$$

where  $Z$  is a standard complex Brownian motion. To verify this one simply computes the generator of  $X$  and checks that it equals  $\Delta_{\mathbb{H}^2}$ , see Section 2.3 in [28].

The last and perhaps most intuitive construction of the Brownian motion in  $\mathbb{H}^2$  is given in Example 3.3.3 on p.84 in [29] which shows that it can be seen as a time-changed 2-dimensional Brownian in  $\mathbb{D}$  stopped upon hitting  $\partial\mathbb{D}$ . For clarity we give the time-change explicitly.

Let  $X_t = r_t e^{i\theta_t}$  be the hyperbolic Brownian motion on  $\mathbb{D}$  and define  $\tau(t)$  to be the inverse of

$$\zeta(t) = \int_0^t \left( \frac{1 - r_s^2}{2} \right)^2 ds,$$

that is

$$\zeta(\tau(t)) = \int_0^{\tau_t} \left( \frac{1-r_s^2}{2} \right)^2 ds = t.$$

Then  $X_{\tau(t)}$  has the same distribution as a 2-dimensional Brownian motion in the unit disk stopped upon hitting the boundary.

We can then define the law of the Brownian motion started at  $z \in \mathbb{H}^2$  as the measure on the subspace of all trajectories  $w \in C(\mathbb{R}_+; \mathbb{H}^2)$  with starting point  $w(0) = z$  which we denote by  $P_z^{\mathbb{H}}$ .

In complete analogy with  $\mathbb{R}^d$  we introduce the energy functional:

$$E_K^{\mathbb{H}^2}(\lambda) = \int_{K \times K} G_{\mathbb{H}^2}(x, y) \lambda(dx) \lambda(dy), \quad (2.27)$$

$$(2.28)$$

where  $\text{supp}(\lambda) \subseteq K$ , and define the hyperbolic capacity of a set  $K \Subset \mathbb{H}^2$  as:

$$\text{cap}_{\mathbb{H}^2}(K) := \left( \inf_{\lambda \in \mathcal{P}(K)} \{E_K^{\mathbb{H}^2}(\lambda)\} \right)^{-1}. \quad (2.29)$$

Moreover, from [12] it is known that

$$\text{cap}_{\mathbb{H}^2}(B(0, r)) = \frac{2\pi}{\log(\coth(r/2))}. \quad (2.30)$$

## 2.6 Brownian excursions in the unit disc

The Brownian excursion measure in the open unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  is a  $\sigma$ -finite measure on trajectories who spend their life time in the unit disk with endpoints on the boundary  $\partial\mathbb{D}$ , see for instance [30], [31], [32] and [33]. Moreover, the Brownian excursion measure is the limit of the so called simple random walk excursion measure, see [34]. Let

$$W_{\mathbb{D}} := \{w \in C([0, T_w], \bar{\mathbb{D}}) : w(0), w(T_w) \in \partial\mathbb{D}, w(t) \in \mathbb{D}, \forall t \in (0, T_w)\}$$

and let  $X_t(w) = w(t)$  be the canonical process on  $W_{\mathbb{D}}$ . We equip  $W_{\mathbb{D}}$  with the metric  $d_{\mathbb{D}}$  defined by

$$d_{\mathbb{D}}(w, w') := |T_w - T_{w'}| + \sup_{s \in [0, 1]} |w(T_w s) - w'(T_{w'} s)|, \quad w, w' \in W_{\mathbb{D}}.$$

It is known that  $W_{\mathbb{D}}$  equipped with  $d_{\mathbb{D}}$  is a complete metric space, see Section 5.1 of [32]. Let  $\mathcal{W}_{\mathbb{D}}$  be the Borel  $\sigma$ -algebra generated by  $d_{\mathbb{D}}$ . Moreover, for  $K \Subset \mathbb{D}$  we let  $W_{K, \mathbb{D}}$  be the set of trajectories in  $W_{\mathbb{D}}$  that hit  $K$ . Let

$$\Omega_{\mathbb{D}} = \left\{ \omega = \sum_{i \geq 0} \delta(w_i, \alpha_i) : (w_i, \alpha_i) \in W_{\mathbb{D}} \times [0, \infty), \omega(W_{K, \mathbb{D}} \times [0, \alpha]) < \infty, \forall K \Subset \mathbb{D}, \alpha \geq 0 \right\}.$$

We endow  $\Omega_{\mathbb{D}}$  with the  $\sigma$ -algebra  $\mathcal{M}_{\mathbb{D}}$  generated by the evaluation maps

$$\omega \mapsto \omega(B), B \in \mathcal{W}_{\mathbb{D}} \otimes \mathcal{B}(\mathbb{R}_+).$$

For a probability measure  $\sigma$  on  $\mathbb{D}$ , denote by  $P_{\sigma}$  the law of Brownian motion with starting point chosen at random according to  $\sigma$ , stopped upon hitting  $\partial\mathbb{D}$ . (Note that  $P_{\sigma}$  has a different meaning if it occurs in a section concerning Brownian inter-lacements.) For  $r > 0$ , let  $\sigma_r$  be the uniform probability measure on  $\partial B(0, r) \subset \mathbb{R}^2$ . The Brownian excursion measure on  $\mathbb{D}$  is defined as the limit

$$\mu = \lim_{\epsilon \rightarrow 0} \frac{2\pi}{\epsilon} P_{\sigma_{1-\epsilon}}, \quad (2.31)$$

which is understood in the following way. Let  $\mathcal{M}(W_{\mathbb{D}})$  be the space of finite measures on  $(W_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}})$  and let  $d$  be the Prokhorov metric,

$$d(\lambda, \eta) = \inf \{ \epsilon > 0 : \lambda(V) \leq \eta(V^{\epsilon}) + \epsilon, \eta(V) \leq \lambda(V^{\epsilon}) + \epsilon, \forall V \in \mathcal{W}_{\mathbb{D}} \},$$

for  $\lambda, \eta \in \mathcal{M}(W_{\mathbb{D}})$ . Then, for any sequence of increasing compact sets in  $\mathbb{D}$ ,  $\{K_n\}_{n \geq 0}$ , such that

$$\bigcup_{n \geq 0} K_n = \mathbb{D},$$

the limits (with respect to  $d$ )

$$\mu^{(n)}(\cdot) = \lim_{\epsilon \rightarrow 0} \frac{2\pi}{\epsilon} P_{\sigma_{1-\epsilon}}(\cdot \cap W_{K_n, \mathbb{D}})$$

form a consistent family of finite measures on  $W_{\mathbb{D}}$ . This means that  $\mu^{(n)}$  satisfies the restriction property

$$\mu^{(n)}(\cdot \cap W_{K_m, \mathbb{D}}) = \mu^{(m)}(\cdot)$$

whenever  $m \leq n$ . We then define  $\mu$  as the sigma-finite measure on  $W_{\mathbb{D}}$  with infinite mass satisfying

$$\mu(\cdot \cap W_{K_n, \mathbb{D}}) = \mu^{(n)}, \forall n \geq 0,$$

see Chapter 5 in [32] or p.299-302 in [30] for more details.

As in [35] we can then define the Brownian excursion process as a Poisson point process on  $W_{\mathbb{D}} \times \mathbb{R}_+$  with intensity measure  $\mu \otimes d\alpha$  and we let  $\mathbb{P}_{\mathbb{D}}$  denote the probability measure corresponding to this process.

For  $\alpha > 0$  and  $\omega = \sum_{i \geq 0} \delta_{w_i, \alpha_i} \in \Omega_{\mathbb{D}}$  we write

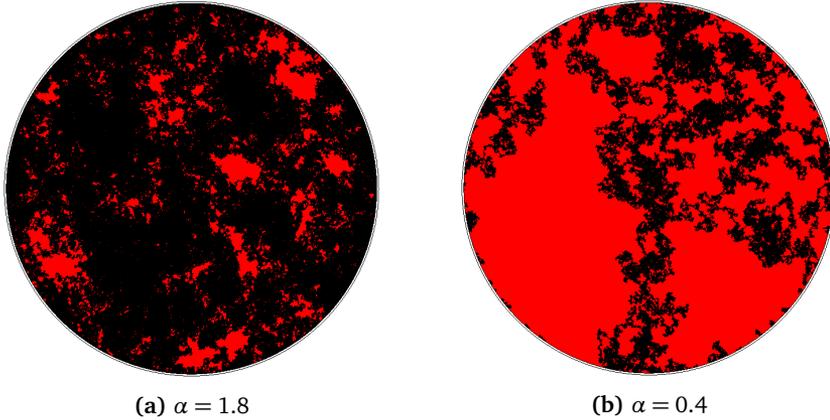
$$\omega_{\alpha} := \sum_{i \geq 0} \delta_{(w_i, \alpha_i)} \mathbf{1}\{\alpha_i \leq \alpha\}, \quad (2.32)$$

and note that under  $\mathbb{P}_{\mathbb{D}}$  the process  $\omega_{\alpha}$  is a Poisson process with intensity measure  $\alpha\mu$ . For  $\alpha > 0$ , the Brownian excursion set at level  $\alpha$  is then defined as

$$\text{BE}_{\alpha}(\omega) := \bigcup_{\alpha_i \leq \alpha} \bigcup_{s \geq 0} w_i(s), \quad \omega = \sum_{i \geq 0} \delta_{(w_i, \alpha_i)} \in \Omega_{\mathbb{D}}, \quad (2.33)$$

and we let  $\mathcal{V}_{\alpha} = \mathbb{D} \setminus \text{BE}_{\alpha}$  denote the vacant set.

Proposition 5.8 in [32] says that  $\mu$ , and consequently  $\mathbb{P}_{\mathbb{D}}$ , are invariant under the conformal automorphisms of  $\mathbb{D}$ .



**Figure 2.2:** A simulation of the random set  $\text{BE} \cap B(0.99)$  for different values of  $\alpha$ .

From [12] we also know that the  $\mu$ -measure of the set of trajectories that hits the ball  $B(0, r_e)$  where

$$r_e = \frac{e^r - 1}{e^r + 1}, \quad r > 0$$

is equal to the hyperbolic capacity of the ball :

$$\mu(W_{B(0, r_e), \mathbb{D}}) = \text{cap}_{\mathbb{H}^2}(B(0, r)) = \frac{2\pi}{\log(\coth(r/2))}.$$

Here  $r_e$  denotes the euclidean radius and  $r$  is the corresponding hyperbolic radius.



# 3 Visibility in the Brownian interlacements

## 3.1 Long range dependence and visibility in the Brownian interlacements

In this section we first study that the 2-point correlation function for the Brownian interlacement set.

First we introduce some notation. For  $z \in \mathbb{R}^d$  define

$$A_z := \{z \in \text{Bl}(\alpha, \rho)\} = \left\{ \omega \in \Omega : \omega \left( W_{B(z, \rho)}^* \times [0, \alpha] \right) \geq 1 \right\},$$

and note that  $\mathbb{P}(A_z)$  is constant with respect to  $z \in \mathbb{R}^d$ . Let  $\alpha > 0$  be fixed and for brevity we define

$$W_z = W_{B(z, \rho)}^* \times [0, \alpha].$$

Then we have the following proposition.

**Proposition 3.1.1.** *For any  $x, y \in \mathbb{R}^d$*

$$\begin{aligned} & \text{Cov} \left( 1 \{x \in \text{Bl}_\alpha^\rho\}, 1 \{y \in \text{Bl}_\alpha^\rho\} \right) \\ &= \left( \frac{1}{\mathbb{P}(\omega(W_x \cap W_y) = 0)} - 1 \right) (1 - \mathbb{P}(A_0))^2. \end{aligned} \tag{3.1}$$

*Proof.* The proof is rather straight-forward. We have

$$\begin{aligned} \mathbb{P}(\{x, y\} \in \text{Bl}) &= \mathbb{P}(A_x \cap A_y) = \mathbb{P}(A_x) + \mathbb{P}(A_y) - \mathbb{P}(A_x \cup A_y) \\ &= \mathbb{P}(A_x) + \mathbb{P}(A_y) - \left( 1 - \mathbb{P}(A_x^c \cap A_y^c) \right) \\ &= 2\mathbb{P}(A_0) - \left( 1 - \mathbb{P}(A_x^c \cap A_y^c) \right). \end{aligned} \tag{3.2}$$

By definition we have

$$\begin{aligned} \mathbb{P}(A_x^c \cap A_y^c) &= \mathbb{P}(\omega(W_x) = 0, \omega(W_y) = 0) \\ &= \mathbb{P}(\omega(W_x \setminus W_y) = 0, \omega(W_y \setminus W_x) = 0, \omega(W_x \cap W_y) = 0) \\ &= \mathbb{P}(\omega(W_x \setminus W_y) = 0) \mathbb{P}(\omega(W_y \setminus W_x) = 0) \mathbb{P}(\omega(W_x \cap W_y) = 0), \end{aligned} \quad (3.3)$$

using independence in the last equality. Since

$$\omega(W_x) = \omega(W_x \setminus W_y) + \omega(W_x \cap W_y)$$

we get

$$\mathbb{P}(A_x^c) = \mathbb{P}(\omega(W_x) = 0) = \mathbb{P}(\omega(W_x \setminus W_y) = 0) \mathbb{P}(\omega(W_x \cap W_y) = 0)$$

which implies that

$$\mathbb{P}(\omega(W_x \setminus W_y) = 0) = \frac{\mathbb{P}(\omega(W_x) = 0)}{\mathbb{P}(\omega(W_x \cap W_y) = 0)}. \quad (3.4)$$

Completely analogously we obtain

$$\mathbb{P}(\omega(W_y \setminus W_x) = 0) = \frac{\mathbb{P}(\omega(W_y) = 0)}{\mathbb{P}(\omega(W_x \cap W_y) = 0)}, \quad (3.5)$$

which, together with Equation (3.3), implies that

$$\mathbb{P}(A_x^c \cap A_y^c) = \mathbb{P}(A_x^c) \mathbb{P}(A_y^c) \frac{1}{\mathbb{P}(\omega(W_x \cap W_y) = 0)}. \quad (3.6)$$

Since

$$\mathbf{Cov}(1\{x \in \text{Bl}_\alpha^\rho\}, 1\{y \in \text{Bl}_\alpha^\rho\}) = \mathbb{P}(A_x \cap A_y) - \mathbb{P}(A_0)^2$$

we get, after some algebraic simplifications using (3.2) and (3.6)

$$\begin{aligned} &\mathbf{Cov}(1\{x \in \text{Bl}_\alpha^\rho\}, 1\{y \in \text{Bl}_\alpha^\rho\}) \\ &= \left( \frac{1}{\mathbb{P}(\omega(W_x \cap W_y) = 0)} - 1 \right) (1 - \mathbb{P}(A_0))^2, \end{aligned}$$

which completes the proof  $\square$

*Remark 3.1.* By Lemma 2.1 in [11] there exists constants  $c(\rho)$ ,  $c'(\alpha, \rho)$  and  $c''(\alpha, \rho)$  such that

$$e^{-c'/(\|x-y\|^{d-2+4\rho})} \leq \mathbb{P}(\omega(W_x \cap W_y) = 0) \leq e^{-c''/\|x-y\|^{d-2+4\rho}}$$

whenever  $\|x-y\| \geq c(\rho)$ . This implies that for  $x, y \in \mathbb{R}^d$  such that  $\|x-y\|_2 \geq c(\rho)$  we can estimate the covariance (Eq. (3.1)) as

$$\frac{c'(\alpha, \rho)}{\|x-y\|^{d-2}} \leq \mathbf{Cov}(1\{x \in \text{Bl}_\alpha^\rho\}, 1\{y \in \text{Bl}_\alpha^\rho\}) \leq \frac{c''(\alpha, \rho)}{\|x-y\|_2^{d-2}}.$$

Now we turn to the question of visibility *inside* the interlacement set. Let  $x \in \partial B(0, 1)$  and denote by  $[0, rx]$  the line-segment between the points 0 and  $rx$  and define

$$f(r) = \mathbb{P}([0, rx] \subset \text{Bl}_\alpha^\rho). \quad (3.7)$$

Note that  $f$  is independent of  $x$  since the law of  $\text{Bl}_\alpha^\rho$  is invariant under the isometries of  $\mathbb{R}^d$ . To derive an upper bound on  $f(r)$  we restrict ourselves to  $d \geq 4$ . The idea is that we want to bound the probability of a Brownian motion hitting  $k$  distinct balls out of  $n$  possible. First we shall need a general estimate on hitting probabilities.

Let  $M_z(x, y) := G(x, y)/G(z, y)$  be the Martin kernel and define

$$\text{cap}_{M_z}(K) := \left( \inf_{\lambda \in \mathcal{P}(K)} \int_{K \times K} M_z(x, y) \lambda(dx) \lambda(dy) \right)^{-1}, \quad (3.8)$$

for  $K$  compact. Proposition 1.1 in [36] states that

$$\frac{1}{2} \text{cap}_{M_z}(K) \leq P_z(H_K < \infty) \leq \text{cap}_{M_z}(K)$$

where the constants 1/2 and 1 are sharp. Using this we can prove the following lemma.

**Lemma 3.1.1.** *Let  $K \Subset \mathbb{R}^d$  and  $x \notin K$  then*

$$\frac{\text{cap}(K)}{2c_d(\text{dist}(x, K) + \text{diam}(K))^{d-2}} \leq P_x(H_K < \infty) \leq \frac{\text{cap}(K)}{c_d \text{dist}(x, K)^{d-2}}, \quad (3.9)$$

where  $c_d$  is given in Equation (2.8).

*Proof.* It suffices to prove that

$$\frac{\text{cap}(K)}{c_d(\text{dist}(z, K) + \text{diam}(K))^{d-2}} \leq \text{cap}_{M_z}(K) \leq \frac{\text{cap}(K)}{c_d \text{dist}(z, K)^{d-2}}.$$

Observing that for  $z \in \mathbb{R}^d$  and  $y \in K \Subset \mathbb{R}^d$  we have

$$\text{dist}(z, K) \leq \|z - y\|_2 \leq \text{dist}(z, K) + \text{diam}(K)$$

which implies that, using Equation (2.8),

$$\frac{\text{dist}(z, K)^{d-2}}{c_d} G(x, y) \leq M_z(x, y) \leq \frac{(\text{dist}(z, K) + \text{diam}(K))^{d-2}}{c_d} G(x, y)$$

Hence for  $\lambda \in \mathcal{P}(K)$  we have the lower

$$\int_{K \times K} M_z(x, y) \lambda(dx) \lambda(dy) \geq \frac{\text{dist}(z, K)^{d-2}}{c_d} \int_{K \times K} G(x, y) \lambda(dx) \lambda(dy),$$

and upper bound

$$\int_{K \times K} M_z(x, y) \lambda(dx) \lambda(dy) \leq \frac{(\text{dist}(z, K) + \text{diam}(K))^{d-2}}{c_d} \int_{K \times K} G(x, y) \lambda(dx) \lambda(dy),$$

which, in view of (3.8), concludes the proof.  $\square$

We shall need some additional notation before the next lemma. Let  $n \in \mathbb{N}$  be non-zero,  $x \in S^{d-1}$ , and let  $x_1, x_2, \dots, x_n \in$  be distinct points placed equidistantly along the line  $[0, rx]$  satisfying

$$\|x_i - x_{i+1}\|_2 := \ell > 2\rho, \forall i \in \{1, 2, \dots, n-1\}, \quad (3.10)$$

so that

$$n \asymp c \lfloor r \rfloor \quad (3.11)$$

for some constant  $c(\rho)$  where  $\lfloor r \rfloor$  denotes the floor of  $r$ . Let  $C = [0, rx]^\rho$  and let  $(X(t))_{t \geq 0}$  be a Brownian motion started according to the normalized equilibrium measure of  $C$ . Let  $B_i = B(x_i, \rho)$ ,  $i = 1, 2, \dots, n$  and let  $Z$  be the number of distinct balls that  $X$  hits. Then we have the following lemma.

**Lemma 3.1.2.** *Assume  $d \geq 4$ . Then there exists a  $L(\rho) > 0$  such that for all  $\ell > L(\rho)$ , there exists a  $p \in (0, 1)$  such that*

$$P_{\tilde{e}_c}(Z = k) \leq p^k, k = 0, 1, 2, \dots, n. \quad (3.12)$$

*Proof.* Recall the definitions of  $X, C$  and  $B_i$  given prior to the statement. Let  $H_i = H_{B_i}$  and denote

$$H_{(1)} = \min_{1 \leq i \leq n} H_i.$$

Moreover, let  $B_{(1)}$  be the ball that  $X$  hits at time  $H_{(1)}$ . Now we define

$$H_{(2)} = \inf \left\{ t > H_{(1)} : X(t) \in \bigcup_{j=1}^n B_j \setminus B_{(1)} \right\}$$

and for  $k \geq 2$  we let

$$H_{(k)} = \inf \left\{ t > H_{(k-1)} : X(t) \in \bigcup_{j=1}^n B_j \setminus \bigcup_{i=1}^k B_{(i)} \right\},$$

where  $B_{(i)}$  is the ball that  $X$  hits at time  $H_{(i)}$ . Now observe that for  $k \geq 1$

$$A_k = \{Z = k\} = \{H_{(k)} < \infty, H_{(k+1)} = \infty\} \subseteq \{H_{(k)} < \infty\}. \quad (3.13)$$

Moreover, we have  $H_{(1)} < \infty$  with a positive probability  $p_1(n) \in (0, 1)$  given by

$$\begin{aligned} p_1 &= P_{\tilde{e}_c}(H_{(1)} < \infty) = P_{\tilde{e}_c} \left( H \left( \bigcup_{i=1}^n B(x_i, \rho) \right) < \infty \right) \\ &= \text{cap} \left( \bigcup_{i=1}^n B(x_i, \rho) \right) / \text{cap}(C) \in (0, 1). \end{aligned}$$

Note that  $p_1(n) < 1$  is uniformly bounded away from 1 in  $n$ , since  $n \asymp r$  by Equations (3.10) and (3.11).

Since  $(\{H_{(k)} < \infty\})_{k \geq 0}$  is a sequence of decreasing events, it follows from Equation (3.13) that it suffices to show that

$$P_{\tilde{e}_c}(H_{(j)} < \infty | H_{(j-1)} < \infty) < p < 1,$$

for some  $p$ . By invoking the Markov property we have

$$P_{\tilde{e}_c}(H_{(i)} < \infty | H_{(i-1)} < \infty) = E_{\tilde{e}_c} \left[ P_{X(H_{(i)})}(H_{(i+1)} < \infty | H_{(i)} < \infty) \right] = E_{\tilde{e}_c} [p_i],$$

where

$$p_i = P_{X(H_{(i)})}(H_{(i+1)} < \infty | H_{(i)} < \infty).$$

So we must show that  $p_i < c < 1$  for all  $i$ . This follows using Equations (3.9) and (3.10) as follows

$$\begin{aligned} P_{X(H_{(i)})}(H_{(i+1)} < \infty | H_{(i)} < \infty) &\stackrel{(3.9)}{\leq} \sum_{l \neq (i)} \frac{c(\rho)}{\|X(H_{(i)}) - x_l\|_2^{d-2}} \\ &\stackrel{(3.10)}{\leq} \frac{c'(\rho)}{\ell^{d-2}} \sum_{l=1}^n \frac{1}{l^{d-2}} \leq c''(\rho)/\ell^{d-2}. \end{aligned}$$

Thus if  $\ell > c''(\rho)$  we see that  $p_i \in [0, c)$  for all  $n \in \mathbb{N}$  and  $i \in \{1, 2, \dots, n\}$  and some  $c < 1$ . So by letting  $p_* = \sup p_i$ , then

$$P_{\tilde{e}_c}(A_k) \leq p_*^k,$$

which concludes the proof. □

*Remark 3.2.* For  $t < -\log(p)$  where  $p$  is as in Lemma 3.1.2 we can conclude that

$$E_{\tilde{e}_c} [e^{tZ}] \leq \frac{1}{1 - e^t p}.$$

We are now ready to prove the main result of this chapter. The proof is inspired by Theorem 2.4 in [8].

**Theorem 3.1.1.** *For  $d \geq 4$ , there exists an  $\alpha^* > 0$  which depends on  $d$  and  $\rho$  such that for  $\alpha < \alpha^*$  and  $r > c(\rho)$*

$$f(r) \leq e^{-ar} \tag{3.14}$$

*for some positive and finite constant  $a = a(\alpha, \rho)$ .*

*Proof.* Let  $C = [0, rx]^\rho$  and let  $\ell$  and  $p \in (0, 1)$  satisfy the conditions of Lemma 3.1.2 and define

$$\chi(t, p) := \frac{1}{1 - e^t p}, t < -\log(p).$$

Let  $B_j$ ,  $1 \leq j \leq N$  be  $N$  independent Brownian motions with initial distribution  $\tilde{e}_C$ . Let  $n$  and  $x_1, x_2, \dots, x_n \in [0, rx]$  satisfy Equations (3.10) and (3.11) and let

$$H(i, j) := H_{B(x_i, \rho)}(B_j), \quad 1 \leq i \leq n, \quad 1 \leq j \leq N.$$

In words,  $H(i, j)$  is the hitting time for the trajectory  $B_j$  of the ball  $B(x_i, \rho)$ . Define

$$\begin{aligned} \xi(i, j) &:= \mathbb{1}\{H(i, j) < \infty\}, \\ \xi_i &:= \sum_{j=1}^N \xi(i, j), \quad \xi^j := \sum_{i=1}^n \xi(i, j), \quad \xi = \sum_{j=1}^N \xi^j = \sum_{i=1}^n \xi_i. \end{aligned} \quad (3.15)$$

Note that  $\xi_i$  describes the number of trajectories that hit  $B(x_i, \rho)$  and  $\xi^j$  describes the number of balls from  $(B(x_i, \rho))_{i=1}^n$  that trajectory  $B_j$  hits. Then

$$\mathbb{P}(x_1, \dots, x_n \in \text{Bl} \mid \omega(W_C^* \times [0, \alpha]) = N) = \mathbb{P}(\cap_{i=1}^n \{\xi_i \geq 1\} \mid \omega(W_C^* \times [0, \alpha]) = N).$$

Write  $P_N := \mathbb{P}(\cdot \mid \omega(W_C^* \times [0, \alpha]) = N)$  for brevity. Then, using a Chernoff bound for  $t > 0$ ,

$$P_N(\cap_{i=1}^n \{\xi_i \geq 1\}) \leq P_N(\xi \geq n) \leq e^{-tn} (E_{\bar{e}_C} [e^{tZ}])^N,$$

since  $\xi$  can be written as the sum of the independent identically distributed random variables  $(\xi^j)_{j=1}^N$  which all have the same distribution as  $Z$ , see (3.15). Recall that by Lemma 3.1.2 and Remark 3.2 we have

$$E_{\bar{e}_C} [e^{tZ}] \leq \chi(t, p).$$

Hence we get,

$$\begin{aligned} f(r) &\leq \mathbb{P}(x_1, \dots, x_n \in \text{Bl}) \leq \sum_{N \geq 0} P_N(\xi \geq n) \mathbb{P}(\omega(W_C^*, [0, \alpha]) = N) \\ &\leq e^{-tn} \sum_{N \geq 0} \frac{(\alpha \text{cap}(C) \chi(t, p))^N}{N!} e^{-\alpha \text{cap}(C)} \\ &= e^{-tn} \exp\{-\alpha \text{cap}(C) + \chi(t, p) \alpha \text{cap}(C)\} \\ &= \exp\{-tn + \alpha \text{cap}(C)(\chi(t, p) - 1)\} \leq e^{-tn + \alpha c' n(\chi(t, p) - 1)}, \end{aligned}$$

where we used the fact that  $\text{cap}(C) \asymp r$  and  $r \asymp n$  by Equation (3.11), where  $c' = c'(\rho)$ .

Hence, we choose  $\alpha$  so that

$$\alpha c'(\chi(t, p) - 1) - t < 0.$$

This is equivalent to

$$\alpha < \frac{t}{c'(\chi(t, p) - 1)}. \quad (3.16)$$

Letting

$$\alpha^* := \sup_{t \in (0, -\log(p))} \frac{t}{c'(\chi(t, p) - 1)}. \quad (3.17)$$

we see that for  $r \geq c(\rho)$  and  $\alpha < \alpha^*$  there exists some constant  $a = a(\alpha, \rho)$  such that

$$f(r) \leq e^{-ar}.$$

□

# 4 Percolation in the vacant set of the Brownian excursions

In this chapter we show that the local description of the Brownian excursions set in the unit disk has the same type of forward-law as the Brownian interlacements, see Equation (4.1) and (4.11) and compare with Equation (2.15) and (2.19). We additionally show that this model has a non-trivial phase transition regarding percolation in the vacant set.

First we recall the characterization of the equilibrium measure for Brownian motion in  $\mathbb{D}$  killed upon hitting  $\partial\mathbb{D}$ , see Theorem 8.8 on p.228 in [22].

For  $K \Subset \mathbb{D}$ , the equilibrium measure,  $e_K(dy)$ , is the unique finite measure satisfying

$$P_x(X(L_K) \in A, 0 \leq L_K < H(\partial\mathbb{D})) = \int_A G(x, y) e_K(dy), \quad (4.1)$$

where  $L_K$  is the last exit time of  $K$  and  $H(\partial\mathbb{D})$  is the hitting time of  $\partial\mathbb{D}$  and  $G(x, y)$  is the Greens function (see Equation (2.25)). Recall that,  $P_x$  denotes the law of the Brownian motion started at  $x \in \mathbb{D}$  stopped upon hitting  $\partial\mathbb{D}$ .

## 4.1 Brownian excursions and Brownian interlacements

The key ingredient in proving that the random set

$$BE_\alpha \cap K, \quad K \Subset \mathbb{D},$$

satisfies similar properties to that of the Brownian interlacements model is proving Equation (4.9). The argument for this is essentially based on a classical result by Port and Stone, Theorem 1.10 on p.58 in [24] and Corollary 4.10 on p.77 in [37]. Theorem 1.10 on p.58 in [24] is proved for  $\mathbb{R}^d$  for  $d \geq 3$  but the argument goes through without any major modifications in the case of a Brownian motion in  $\mathbb{D}$  stopped upon hitting  $\partial\mathbb{D}$ . We shall nevertheless give the proof of this for completeness.

Let  $K$  be a compact set in  $\mathbb{D}$  with nonempty interior and smooth boundary and

define the hitting time distribution

$$h_K(y, \cdot) = P_y(X(H_K) \in \cdot, H_K < \infty). \quad (4.2)$$

Moreover, for  $x \in \mathbb{D}$

$$\begin{aligned} P_x(X(t) \in A) &= P_x(X(t) \in A, H_K > t) + P_x(X(t) \in A, H_K < t) \\ &= \int_A p_K(t, x, y) dy + E_x \left[ \int_A p(t - H_K, X(H_K), y) dy 1\{H_K < t\} \right] \\ &= \int_A p_K(t, x, y) dy + \int_A r_K(t, x, y) dy \end{aligned}$$

where  $p_K(t, x, y)$  denotes the transition density of the Brownian motion killed upon hitting  $K$  and

$$r_K(t, x, y) := E_x[p(t - H_K, X(H_K), y) 1\{H_K < t\}].$$

Furthermore, define the  $\lambda$ -potentials

$$G^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) dt, \quad G_K^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_K(t, x, y) dt,$$

and note that, since  $p(t, x, y)$  and  $p_K(t, x, y)$  are symmetric functions,  $G^\lambda$  and  $G_K^\lambda$  are symmetric.

Completely analogously to  $\mathbb{R}^d$  we have that Equation (1) on p.41 in [24] holds for  $\mathbb{D}$  which states that

$$G^\lambda(x, y) = G_K^\lambda(x, y) + \int G^\lambda(z, y) h_K^\lambda(x, dz), \quad (4.3)$$

where

$$h_K^\lambda(x, A) = E_x[e^{-\lambda H_K} 1\{H_K < \infty, X(H_K) \in A\}]. \quad (4.4)$$

Then by letting  $\lambda \rightarrow 0$  and applying the monotone convergence theorem to  $G^\lambda, G_K^\lambda$  and using the definition of  $h_K^\lambda$  we obtain the *fundamental identity*,

$$G(x, y) = G_K(x, y) + \int G(z, y) h_K(x, dz) \quad (4.5)$$

see Equation (2) on p.55 in [24]. Using that  $G$  and  $G_K$  are symmetric, Equation (4.5) implies that the following holds

$$\int G(z, y) h_K(x, dz) = \int G(z, x) h_K(y, dz), \quad (4.6)$$

which is identical to Equation (4) on p.58 in [24]. Now Corollary 4.10 on p.77 in [37] states that a finite measure  $\mu$  supported on  $\partial K$  is the equilibrium measure if

and only if

$$\int G(x, y)\mu(dy) = 1, \quad \forall x \in K. \quad (4.7)$$

Since  $y \mapsto G(x, y)$  is a harmonic function on  $\mathbb{D} \setminus \{x\}$ , see Theorem 3.35 [22] on p.81, we see that for  $K = B(0, r)$ ,  $0 < r < 1$ , we have

$$\int_{\partial B(0, r)} G(x, y)\sigma_r(dy) = \frac{\log(1/r)}{2\pi},$$

where  $\sigma_r$  is the uniform probability measure on  $\partial B(0, r)$ .

Hence, the measure

$$e_r(dy) := \frac{2\pi}{\log(1/r)}\sigma_r(dy) \quad (4.8)$$

is the equilibrium measure for the ball  $B(0, r)$ , and the capacity is given by

$$\text{cap}(B(0, r)) = \frac{2\pi}{\log(1/r)},$$

compare with Equation (2.30) (where the equation is given in hyperbolic radius).

Thus, if we let  $K \Subset B(0, r)$  and define

$$e_K(\cdot) = \int_{\partial B(0, r)} P_x(X(H_K) \in \cdot, H_K < \infty)e_r(dx), \quad (4.9)$$

then using Equation (4.6) one can verify that Equation (4.7) holds. This means that  $e_K(\cdot)$  is the equilibrium measure for  $K$ .

Recalling the definition of the Brownian excursion measure we see that for  $A \in \mathcal{W}_{\mathbb{D}}$  we have, similarly to Remark 2.1.3 on p.568 [10],

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{2\pi}{\epsilon} P_{\sigma_{1-\epsilon}}(\{(X_{t+H_K})_{t \geq 0} \in A\} \cap \{H_K < \infty\}) \\ & \stackrel{(4.8)}{=} \lim_{\epsilon \rightarrow 0} \frac{-\log(1-\epsilon)}{\epsilon} E_{e_{1-\epsilon}}[P_{X(H_K)}(A)1\{H_K < \infty\}] \\ & = \lim_{\epsilon \rightarrow 0} \frac{-\log(1-\epsilon)}{\epsilon} \int_{\partial B(0, 1-\epsilon)} e_{1-\epsilon}(dx) \int_{\partial K} h_K(x, dy) P_y(A) \\ & = \lim_{\epsilon \rightarrow 0} \frac{-\log(1-\epsilon)}{\epsilon} \int_{\partial K} P_y(A) \int_{\partial B(0, 1-\epsilon)} e_{1-\epsilon}(dx) h_K(x, dy) \\ & \stackrel{(4.9)}{=} \int_{\partial K} e_K(dy) P_y(A) = P_{e_K}(A). \end{aligned} \quad (4.10)$$

In words, this means the following. The local description of the Brownian excursions set satisfies the following distributional equality. For  $K \Subset \mathbb{D}$ , let  $N_K \sim \text{Po}(\alpha \text{cap}(K))$  and let  $\tilde{e}_K$  be the normalized equilibrium measure on  $K$ . Condition-

ally on  $N_K$ , let  $(B_i)_{i=1}^{N_K}$  be a collection of independent Brownian motions in the unit disk with initial distribution  $B_i(0) \sim \tilde{e}_K$ . Then following distributional equality holds

$$\text{BE}_\alpha \cap K \stackrel{d}{=} \bigcup_{i=1}^{N_K} [B_i] \cap K \quad (4.11)$$

where  $[B_i]$  denotes the trace of the Brownian motion.

## 4.2 Percolation in the vacant set

In this section we equip the unit disk with the hyperbolic metric, that is we work in the Poincaré disk model of  $\mathbb{H}^2$ .

Here we prove that the vacant set of the Brownian excursions model undergoes a phase transition concerning percolation. Note, that by construction  $(\mathcal{V}_\alpha)_{\alpha \geq 0}$  is a sequence of decreasing random sets in  $\alpha$ . We say that  $\mathcal{V}_\alpha$  percolates if  $\mathcal{V}_\alpha$  contains an unbounded connected component and let

$$\alpha_* = \sup \{ \alpha > 0 : \mathbb{P}_{\mathbb{D}}(\mathcal{V}_\alpha \text{ percolates}) > 0 \}. \quad (4.12)$$

In what follows, we will often write  $\mathcal{V}$  instead of  $\mathcal{V}_\alpha$ . Furthermore, for use below, we say that we have visibility to infinity in the random set  $\mathcal{S}$  from the origin, if there is some  $\theta \in [0, 2\pi)$  such that  $[0, e^{i\theta}) \subset \mathcal{S}$ , where  $[0, e^{i\theta})$  denotes the line-segment from 0 to  $e^{i\theta}$  which in the hyperbolic metric is an infinite half-line.

The main result of this section is the following theorem.

**Theorem 4.2.1.** *The critical value  $\alpha_*$  for  $\mathcal{V}_\alpha$  is non-trivial and satisfies*

$$\alpha_* \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right) \quad (4.13)$$

The lower bound follows from the fact that for  $\alpha < \frac{\pi}{4}$  there is visibility to infinity in  $\mathcal{V}_\alpha$ , see Theorem 2.3 in [12] on p.9.

In the proof of Theorem 4.2.1 we will use a Poisson line process induced by the endpoints of the trajectories in the Brownian excursion process. Recall that the geodesic between the points  $x, y \in \mathbb{D}$  is a segment of the circle that passes through  $x$  and  $y$  and intersects the boundary of  $\partial\mathbb{D}$  orthogonally. To construct this line process we will use a different description of the Brownian excursion measure than earlier used in the thesis. Let  $z = e^{i\theta}, w = e^{i\theta'} \in \partial\mathbb{D}$  and let  $H_{\mathbb{D}}$  be the so called boundary Poisson kernel given by

$$H_{\mathbb{D}}(z, w) := \frac{1}{2\pi} \frac{1}{1 - \cos(\theta - \theta')}.$$

Then the Brownian excursion measure is given by

$$\mu_{\mathbb{D}} = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} H_{\mathbb{D}}(z, w) \mu_{\mathbb{D}}(z, w) |dz| |dw|,$$

where  $\mu_{\mathbb{D}}(z, w)$  is a Brownian probability measure derived in [38] on p. 126. Formally,  $\mu_{\mathbb{D}}(z, w)$  is the law of a Brownian motion started at  $z$  conditioned on exiting  $\mathbb{D}$  for the first time at  $w$ . The equivalence between this definition and Equation (2.31) is explained on p. 127-128 in [38].

Now we give the definition of the *canonical* Poisson line process on  $\mathbb{H}^2$ . Let  $\mathbb{G}$  be the Grassmannian of unoriented lines in  $\mathbb{H}^2$  and let  $\lambda$  be the unique (up to scaling)  $\sigma$ -finite measure that is invariant under the induced isometries from  $\mathbb{H}^2$ , see Chapter 17 in [39] regarding the details. Let  $A, B \subset \partial\mathbb{H}^2 \equiv \partial\mathbb{D}$  be arcs on the boundary and let  $\mathcal{L}_{A,B}$  be all lines in  $\mathbb{G}$  with one endpoint in  $A$  and  $B$  respectively. It is a well-known fact that  $\mathbb{G}$  can be identified with the set of all unordered pairs of points on the boundary of  $\mathbb{H}^2$ ,

$$\mathbb{M} := \{\{x, y\} : x, y \in \partial\mathbb{D}, x \neq y\},$$

see [40]. The invariant measure is given by

$$\lambda(\mathcal{L}_{A,B}) = \frac{1}{2} \int_A \int_B \frac{1}{1 - \cos(\theta - \theta')} d\theta d\theta',$$

where we used the same normalization as in [40]. The Poisson line process at level  $\alpha$  on  $\mathbb{H}^2$  is then defined as a Poisson process with intensity measure  $\alpha\lambda$ .

We now construct the line process induced by the Brownian excursions process. For each  $w \in W_{\mathbb{D}}$  let  $g(w)$  be the unique unoriented line in  $\mathbb{G}$  with endpoints  $w(0), w(T_w) \in \partial\mathbb{D}$ . Define the measure  $\gamma$  on  $\mathbb{G}$  by

$$\begin{aligned} \gamma(\mathcal{L}_{A,B}) &= \mu_{\mathbb{D}}(\{w : g(w) \in \mathcal{L}_{A,B}\}) \\ &= \int_{A \times B} H_{\mathbb{D}}(\theta, \theta') d\theta d\theta' + \int_{B \times A} H_{\mathbb{D}}(\theta, \theta') d\theta d\theta' \\ &= \frac{1}{\pi} \int_{A \times B} \frac{1}{1 - \cos(\theta - \theta')} d\theta d\theta' = \frac{2}{\pi} \lambda(\mathcal{L}_{A,B}). \end{aligned}$$

Recalling the definition of  $\omega_{\alpha}$ , see Equation (2.32), we then define

$$L_{\alpha}(\omega) = \sum_{w \in \text{supp } \omega_{\alpha}} \delta_{g(w)}, \quad (4.14)$$

and note that under  $\mathbb{P}_{\mathbb{D}}$

$$L_{\alpha} \text{ is a Poisson line process with intensity measure } \frac{2}{\pi} \alpha\lambda, \quad (4.15)$$

where we recall that  $\lambda$  is the intensity measure of the canonical Poisson line process on  $\mathbb{G}$ .

Now, let

$$\mathcal{L} = \bigcup_{\ell \in \text{supp } L_\alpha} \ell, \quad (4.16)$$

and

$$\mathcal{Z} = \mathbb{H}^2 \setminus \mathcal{L}. \quad (4.17)$$

We shall need the following facts about  $\mathcal{L}$  and  $\mathcal{Z}$ , which are contained in Section 6 of [40].

Let  $\alpha_{\mathcal{L}} = \pi/2$  be the critical value for visibility to infinity in  $\mathcal{Z}$ . Then, if  $\alpha \geq \alpha_{\mathcal{L}}$  almost surely there is no visibility to infinity from the origin in  $\mathcal{Z}$  (4.18)

and

$$\mathcal{Z} \text{ does not percolate.} \quad (4.19)$$

If  $\alpha < \alpha_{\mathcal{L}}$ , then

with positive probability there is visibility to infinity from the origin in  $\mathcal{Z}$  (4.20)

and

$$\mathcal{Z} \text{ percolates almost surely.} \quad (4.21)$$

From now on, for a random closed set  $S$  in  $\mathbb{H}^2$  let

$$\text{Perc}_S = \left\{ 0 \overset{S}{\longleftrightarrow} \infty \right\} \quad (4.22)$$

be the event that there is an infinite component of  $S$  containing the origin.

In the proof of Theorem 4.2.1 it turns out that there is a certain subset of trajectories that complicate matters and we now define the set of unproblematic trajectories. For  $w \in W_{\mathbb{D}}$  and let  $\text{arc}(w)$  be the arc on  $\partial\mathbb{D}$  that is shadowed by  $w$ :

$$\text{arc}(w) = \{e^{i\theta} : [0, e^{i\theta}) \cap w \neq \emptyset\}, \quad (4.23)$$

where  $[0, e^{i\theta})$  denotes the line segment from 0 to  $e^{i\theta}$ . Let  $\Theta(w) = \text{length}(\text{arc}(w))$  and let

$$A_\theta = \{w \in W_{\mathbb{D}} : \Theta(w) \geq \theta\}$$

denote all trajectories whose arc “shadows” more than a segment of length  $\theta$  of the boundary. Also, let  $\text{arc}_i(w)$  denote the closed arc of length  $\leq \pi$  which has  $w(0)$  and  $w(T_w)$  as endpoints and let  $\Theta_i(w) = \text{length}(\text{arc}_i(w))$ . Note that with probability 0 there exists some  $w$  with  $\Theta_i(w) = \pi$ .

From Lemma 5.2 in [12] we know that under  $\mathbb{P}_{\mathbb{D}}$  the random variable  $\omega_\alpha(A_\theta)$  is a Poisson distributed random variable with parameter  $\alpha\mu(A_\theta) = 8\alpha/\theta$ . Let

$$C_0 = \{w \in W_{\mathbb{D}} : \Theta(w) < \pi\},$$

and crucially note that if  $w \in C_0$  then any continuous curve from 0 to  $\text{arc}_i(w)$  must cross  $w$ . Moreover, since

$$\mu(C_0^c) = \frac{8}{\pi}, \quad (4.24)$$

we see that under  $\mathbb{P}_{\mathbb{D}}$  the random variable  $\omega(C_0^c)$  satisfies

$$\omega(C_0^c) \sim \text{Po}(\alpha 8/\pi).$$

Note that if  $\omega(C_0^c) = 0$  then non-percolation in  $\mathcal{Z}$  implies non-percolation in  $\mathcal{V}$ .

*Proof of Theorem 4.2.1.* We already mentioned that by Theorem 2.3 on p.9 in [12] we have that  $\alpha_* \geq \pi/4$ . However we can get a possibly different lower bound by using the following argument.

By Theorem 1.1 p.324 in [40] there is a universal constant  $0 < p_0(\mathbb{H}^2) < 1$  such that if a random closed set,  $Z$ , in  $\mathbb{H}^2$  which is invariant under isometries and satisfies  $\mathbb{P}(B(o, 1) \subset Z) > p_0$  then this random set contains hyperbolic lines with a positive probability. Thus, for  $\epsilon > 0$  let

$$\mathcal{V}^\epsilon := \overline{\mathbb{H}^2} \setminus \overline{\text{BE}_\alpha^\epsilon}.$$

Then we have

$$\mathbb{P}_{\mathbb{D}}(B(0, 1) \subset \mathcal{V}^\epsilon) = e^{-\alpha \text{cap}_{\mathbb{H}^2}(B(0, 1+\epsilon))}.$$

Hence if  $p_0 < e^{-\alpha \text{cap}_{\mathbb{H}^2}(B(0, 1+\epsilon))}$  then  $\mathcal{V}^\epsilon$  contains hyperbolic lines (and therefore percolates) with positive probability. Rearranging the inequality and since  $\epsilon$  is arbitrary, this yields the following lower bound

$$\alpha_c \geq -\frac{\log(p_0)}{\text{cap}_{\mathbb{H}^2}(B(0, 1))} = -\log(p_0) \frac{\log(\coth(1/2))}{2\pi}. \quad (4.25)$$

Since we do not know the value of  $p_0$  we do not know whether this bound is better than  $\pi/4$  or not.

To show that  $\alpha_c < \infty$  we shall use the line process  $L(\omega)$  defined in Equation (4.16). By Equation (4.24) we know that  $C_0^c$  is a finite set almost surely. Hence, for  $\epsilon > 0$  let  $\theta_\epsilon > 0$  be so small that

$$\mathbb{P}_{\mathbb{D}}(C_0^c \subseteq \{w \in W_{\mathbb{D}} : \Theta_i(w) > \theta_\epsilon\}) \geq 1 - \epsilon, \quad (4.26)$$

and let

$$\omega_{\alpha, \epsilon} = \omega_\alpha \mathbf{1}_{\{w \in W_{\mathbb{D}} : \Theta_i(w) \leq \theta_\epsilon\}}.$$

For  $\alpha \geq \pi/2$ , we have by Proposition 6.1 in [40] that  $\mathbb{P}_{\mathbb{D}}(\omega_\alpha \in \text{Perc}_{\mathcal{Z}}) = 0$ . Since

$$\mathbb{P}_{\mathbb{D}}(\omega_\alpha \in \text{Perc}_{\mathcal{Z}}) \geq \mathbb{P}_{\mathbb{D}}(\omega_{\alpha, \epsilon} \in \text{Perc}_{\mathcal{Z}} \mid \omega_\alpha = \omega_{\alpha, \epsilon}) \mathbb{P}_{\mathbb{D}}(\omega_\alpha = \omega_{\alpha, \epsilon})$$

and  $\mathbb{P}_{\mathbb{D}}(\omega_\alpha = \omega_{\alpha, \epsilon}) > 0$  we must have

$$\mathbb{P}_{\mathbb{D}}(\omega_{\alpha, \epsilon} \in \text{Perc}_{\mathcal{Z}}) = 0, \quad \forall \epsilon > 0. \quad (4.27)$$

Finally, since

$$\{\omega_{\alpha, \epsilon} \in \text{Perc}_{\mathcal{Z}}\}^c \cap \{C_0^c \subset \{w : \Theta_i(w) > \theta_\epsilon\}\} \subset \{\omega_\alpha \in \text{Perc}_{\mathcal{V}}\}^c$$

we see that using (4.26) and (4.27)

$$\mathbb{P}_{\mathbb{D}}(\{\omega_\alpha \in \text{Perc}_{\mathcal{V}}\}^c) \geq 1 - \epsilon$$

for  $\alpha \geq \pi/2$ . Since  $\epsilon$  is arbitrary we get

$$\mathbb{P}_{\mathbb{D}}(\{\omega_\alpha \in \text{Perc}_\nu\}^c) = 1.$$

Hence  $\alpha_c \leq \pi/2$ .

□

# 5 Summary of papers

## 5.1 Paper I

To better understand the Brownian interacements model, we study the problem of visibility in the vacant set  $\mathcal{V}_{\alpha,\rho}$ . The visibility in a fixed direction in  $\mathcal{V}_{\alpha,\rho}$  from a given point  $x \in \mathbb{R}^d$  ( $d \geq 3$ ) is defined as the longest distance you can move from  $x$  in the direction, without hitting  $\text{Bl}_\alpha^\rho$ . The probability that the visibility in a fixed direction from  $x$  is larger than  $r \geq 0$  is denoted by  $f(r) = f(r, \alpha, \rho, d)$ . The visibility from  $x$  is then defined as the longest distance you can move in *some* direction, and the probability that the visibility is larger than  $r \geq 0$  is denoted by  $P_{\text{vis}}(r) = P_{\text{vis}}(r, \alpha, \rho, d)$ .

We show that the visibility in the vacant set  $P_{\text{vis}}(r)$  satisfies the following bounds

$$cr^{d-1}f(r) \leq P_{\text{vis}}(r) \leq Cr^{2(d-1)}f(r)$$

as  $r \rightarrow \infty$ , where  $f(r)$  is the visibility in a fix direction of distance  $r$ .

In addition to the Brownian interacements model, we consider the Brownian excursion model in the unit disk which can be thought of as the 2-dimensional hyperbolic analogue of the Brownian interacements. We show that the Brownian excursion model undergoes a phase transition concerning visibility to infinity in the vacant set and we determine the critical value to  $\alpha_c = \pi/4$  and show that at criticality we have no visibility to infinity, using a classical theorem from Shepp on circle covering.



# Bibliography

- [1] Baxter RJ. Exactly Solved Models in Statistical Mechanics. London: Academic Press; 1982.
- [2] Meester R, Roy R. Continuum percolation. 119. Cambridge University Press; 1996.
- [3] Grimmett G, service) SO, (e-book collection) SA. Percolation. vol. 321. Second;1; ed. Berlin, Heidelberg: Springer Berlin Heidelberg; 1999;1996;.
- [4] Broadbent SR, Hammersley JM. Percolation processes: I. Crystals and mazes. In: Mathematical Proceedings of the Cambridge Philosophical Society. vol. 53. Cambridge University Press; 1957. p. 629–641.
- [5] Erzbischof. Bond percolation with  $p=51$  in a two-dimensional grid of  $50 \times 50$ ; [Accessed 25rd October 2017]. [Online]. Available from: <https://commons.wikimedia.org/w/index.php?curid=655409>.
- [6] Bollobas B, Riordan O. Percolation. Cambridge: Cambridge University Press; 2006.
- [7] Grimmett G, (e-book collection) S. The random-cluster model. vol. 333;333.; 1st ed. Dordrecht;New York;: Springer; 2006.
- [8] Sznitman AS. Vacant set of random interacements and percolation. Annals of mathematics. 2010;p. 2039–2087.
- [9] Sidoravicius V, Sznitman AS. Percolation for the vacant set of random interacements. Comm Pure Appl Math. 2009;62(6):831–858. Available from: <http://dx.doi.org/10.1002/cpa.20267>.
- [10] Sznitman AS. On scaling limits and Brownian interacements. Bulletin of the Brazilian Mathematical Society, New Series. 2013;44(4):555–592.
- [11] Li X. Percolative properties of Brownian interacements and its vacant set. arXiv preprint arXiv:161008204. 2016;.
- [12] Elias O, Tykesson J. Visibility in the vacant set of the Brownian interacements and the Brownian excursion process. arXiv preprint arXiv:170909052. 2017;.
- [13] Harding ER, Kendall DG. Stochastic Geometry. London: John Wiley & Sons; 1974.

- [14] Kallenberg O. Random Measures. vol. 23. Berlin: Akademie-Verlag; 1975.
- [15] Benjamini I, Sznitman AS. Giant component and vacant set for random walk on a discrete torus. *J Eur Math Soc (JEMS)*. 2008;10(1):133–172. Available from: <http://dx.doi.org/10.4171/JEMS/106>.
- [16] Windisch D. The vacant set left by a random walk on a discrete torus;. [Accessed 25rd October 2017]. [Online]. Available from: <https://sites.google.com/site/dwindisch/torus1>.
- [17] Ráth B. A short proof of the phase transition for the vacant set of random interlacements. *Electron Commun Probab*. 2015;20:no. 3, 11. Available from: <http://dx.doi.org/10.1214/ECP.v20-3734>.
- [18] Teixeira A, et al. On the uniqueness of the infinite cluster of the vacant set of random interlacements. *The Annals of Applied Probability*. 2009;19(1):454–466.
- [19] Teixeira A. Interlacement percolation on transient weighted graphs. *Electron J Probab*. 2009;14(54):1604–1628.
- [20] Ráth B, Sapozhnikov A. Connectivity properties of random interlacement and intersection of random walks. *ALEA Lat Am J Probab Math Stat*. 2012;9:67–83.
- [21] Procaccia EB, Tykesson J. Geometry of the random interlacement. *Electron Commun Probab*. 2011;16:528–544. Available from: <http://dx.doi.org/10.1214/ECP.v16-1660>.
- [22] Mörters P, Peres Y. Brownian motion. vol. 30. Cambridge University Press; 2010.
- [23] Brelot M, Gowrisankaran K, Brelot M, Brelot M, Mathematician F. Lectures on potential theory. vol. 19. Tata Institute of Fundamental Research Bombay; 1967.
- [24] Port SC, Stone CJ. Brownian Motion and Classical Potential Theory. New York: Academic Press; 1978.
- [25] Janson S. Bounds on the distributions of extremal values of a scanning process. *Stochastic Process Appl*. 1984;18(2):313–328. Available from: [http://dx.doi.org/10.1016/0304-4149\(84\)90303-X](http://dx.doi.org/10.1016/0304-4149(84)90303-X).
- [26] Cannon JW, Floyd WJ, Kenyon R, Parry WR. Hyperbolic geometry. In: *Flavors of geometry*. vol. 31 of Math. Sci. Res. Inst. Publ. Cambridge Univ. Press, Cambridge; 1997. p. 59–115.
- [27] McKean HP, et al. An upper bound to the spectrum of  $\Delta$  on a manifold of negative curvature. *Journal of Differential Geometry*. 1970;4(3):359–366.
- [28] Valkó B, Virág B. The  $\text{Sine}_\beta$  operator. *Invent Math*. 2017;209(1):275–327. Available from: <http://dx.doi.org/10.1007/s00222-016-0709-x>.

- [29] Hsu EP. Stochastic analysis on manifolds. vol. 38. American Mathematical Soc.; 2002.
- [30] Lawler GF, Werner W. Universality for conformally invariant intersection exponents. *J Eur Math Soc (JEMS)*. 2000;2(4):291–328. Available from: <http://dx.doi.org/10.1007/s100970000024>.
- [31] Virág B. Brownian beads. *Probab Theory Related Fields*. 2003;127(3):367–387. Available from: <http://dx.doi.org/10.1007/s00440-003-0289-8>.
- [32] Lawler GF. Conformally invariant processes in the plane. vol. 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; 2005.
- [33] Lawler GF, Werner W. The Brownian loop soup. *Probab Theory Related Fields*. 2004;128(4):565–588. Available from: <http://dx.doi.org/10.1007/s00440-003-0319-6>.
- [34] Kozdron MJ. On the scaling limit of simple random walk excursion measure in the plane. *ALEA Lat Am J Probab Math Stat*. 2006;2:125–155.
- [35] Wu H. On the occupation times of Brownian excursions and Brownian loops. In: *Séminaire de Probabilités XLIV*. vol. 2046 of *Lecture Notes in Math*. Springer, Heidelberg; 2012. p. 149–166. Available from: [http://dx.doi.org/10.1007/978-3-642-27461-9\\_7](http://dx.doi.org/10.1007/978-3-642-27461-9_7).
- [36] Benjamini I, Pemantle R, Peres Y. Martin capacity for Markov chains. *Ann Probab*. 1995;23(3):1332–1346. Available from: [http://links.jstor.org/sici?sici=0091-1798\(199507\)23:3<1332:MCFMC>2.0.CO;2-I&origin=MSN](http://links.jstor.org/sici?sici=0091-1798(199507)23:3<1332:MCFMC>2.0.CO;2-I&origin=MSN).
- [37] Sznitman AS, service) SO, (e-book collection) SA. *Brownian Motion, Obstacles and Random Media*. 1st ed. Berlin, Heidelberg: Springer Berlin Heidelberg; 1998;1997;.
- [38] Lawler GF. Conformally invariant processes in the plane. 114. American Mathematical Soc.; 2008.
- [39] Santalo LA. *Integral geometry and geometric probability*. London: Addison-Wesley; 1976.
- [40] Benjamini I, Jonasson J, Schramm O, Tykesson J. Visibility to infinity in the hyperbolic plane, despite obstacles. *Alea*. 2009;6:323–342.



# Visibility in the vacant set of the Brownian interlacements and the Brownian excursion process

Olof Elias\*      Johan Tykesson†

September 15, 2017

## Abstract

We consider the Brownian interlacements model in Euclidean space, introduced by A.S. Sznitman in [25]. We give estimates for the asymptotics of the visibility in the vacant set. We also consider visibility inside the vacant set of the Brownian excursion process in the unit disc and show that it undergoes a phase transition regarding visibility to infinity as in [1]. Additionally, we determine the critical value and that there is no visibility to infinity at the critical intensity.

## 1 Introduction

In this paper, we study visibility inside the vacant set of two percolation models; the Brownian interlacements model in  $\mathbb{R}^d$  ( $d \geq 3$ ), and the Brownian excursion process in the unit disc. Below, we first informally discuss Brownian interlacements model and our results for that model, and then we move on the Brownian excursions process.

The Brownian interlacements model is defined as a Poisson point process on the space of doubly infinite continuous trajectories modulo time-shift in  $\mathbb{R}^d$ ,  $d \geq 3$ . The aforementioned trajectories essentially look like the traces of double-sided Brownian motions. It was introduced by A.S Sznitman in [25] as a means to study scaling limits of the occupation measure of continuous time random interlacements on the lattice  $N^{-1}\mathbb{Z}^d$ . The Brownian interlacements model can be considered to be the continuous counterpart of the random interlacements model, which is defined as a Poisson point process on the space of doubly infinite trajectories in  $\mathbb{Z}^d$ ,  $d \geq 3$ , and was introduced in [24]. Both models exhibit infinite range dependence of polynomial decay, which often complicates the application of standard arguments. Random interlacements on  $\mathbb{Z}^d$  have received quite a lot of attention since their introduction. For example, percolation in the vacant set of

---

\*Department of Mathematics, Chalmers University of Technology and Gothenburg University, Sweden. E-mail: olofel@chalmers.se

†Department of Mathematics, Chalmers University of Technology and Gothenburg University, Sweden. E-mail: johant@chalmers.se. Research supported by the Knut and Alice Wallenberg foundation.

the model have been studied in [22] and [24]. Connectivity properties of the interlacement set have been studied in [20], [19], [4] and [10]. For the Brownian interlacements model, percolative and connectivity properties were studied in [14].

We will recall the precise definition of the Brownian interlacement model in Section 2, where we will also give the precise formulation of our main results, but first we discuss our results somewhat informally. In the present work, we study *visibility* inside the vacant set of the Brownian interlacements. For  $\rho > 0$  and  $\alpha > 0$ , the vacant set  $\mathcal{V}_{\alpha,\rho}$  is the complement of the random closed set  $\text{Bl}_\alpha^\rho$ , which is the closed  $\rho$ -neighbourhood of the union of the traces of the trajectories in the underlying Poisson point process in the model. Here  $\alpha$  is a multiplicative constant of the intensity measure (see (9)) of the Poisson point process, governing the amount of trajectories that appear in the process. The visibility in a fixed direction in  $\mathcal{V}_{\alpha,\rho}$  from a given point  $x \in \mathbb{R}^d$  ( $d \geq 3$ ) is defined as the longest distance you can move from  $x$  in the direction, without hitting  $\text{Bl}_\alpha^\rho$ . The probability that the visibility in a fixed direction from  $x$  is larger than  $r \geq 0$  is denoted by  $f(r) = f(r, \alpha, \rho, d)$ . The visibility from  $x$  is then defined as the longest distance you can move in *some* direction, and the probability that the visibility is larger than  $r \geq 0$  is denoted by  $P_{\text{vis}}(r) = P_{\text{vis}}(r, \alpha, \rho, d)$ . Clearly,  $P_{\text{vis}}(r) \geq f(r)$ , but it is of interest to more closely study the relationship between the functions  $P_{\text{vis}}(r)$  and  $f(r)$ . Our main result for Brownian interlacements in  $\mathbb{R}^d$ , Theorem 2.2, gives upper and lower bounds of  $P_{\text{vis}}(r)$  in terms of  $f(r)$ . In particular, Theorem 2.2 show that the rates of decay (in  $r$ ) for the two functions differ with at most a polynomial factor. It is worth mentioning that even if the Brownian interlacements model in some aspects behaves very differently from more standard continuum percolation models like the Poisson Boolean model, when it comes to visibility the difference does not appear to be too big. The proof of Theorem 2.2 uses first and second moment methods and is inspired by the proofs of Lemmas 3.5 and 3.6 of [1]. The existence of long-range dependence in the model creates some extra complications to overcome. It seems to us that the arguments in the proof of Theorem 2.2 are possible to adapt to other percolation models based on Poisson-processes on infinite objects, for example the Poisson cylinder model [28].

We now move on to the Brownian excursion process in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . This process is defined as a Poisson point process on the space of Brownian paths that start and end on  $\partial\mathbb{D}$ , and stay inside  $\mathbb{D}$  in between. The intensity measure is given by  $\alpha\mu$  where  $\mu$  is the Brownian excursion measure (see for example [12], [11]) and  $\alpha > 0$  is a constant. This process was studied in [30], where, among other things, connections to Gaussian free fields were made. The union of the traces of the trajectories in this Poisson point process is a closed random set which we denoted by  $\text{BE}_\alpha$ , and the complement is denoted by  $\mathcal{V}_\alpha$ . Again, we consider visibility inside the vacant set. In Theorem 2.3, we show that, there is a critical level  $\alpha_c = \pi/4$  such that if  $\alpha < \alpha_c$ , with positive probability there is some  $\theta \in [0, 2\pi)$  such that the line-segment  $[0, e^{i\theta}]$  (which has infinite length in the hyperbolic metric) is contained in  $\mathcal{V}_\alpha$ , while if  $\alpha \geq \alpha_c$  the set of such  $\theta$  is a.s. empty. A similar phase transition is known to hold for the Poisson Boolean model of continuum percolation and some other models in the hyperbolic plane, see [1] and [15]. As seen by Theorem 2.2, such a phase transition does not occur for the

set  $\mathcal{V}_{\alpha,\rho}$  in the Brownian interacements model in Euclidean space, when  $\rho > 0$ . The proof of Theorem 2.3 is based on circle covering techniques, using a sharp condition by Shepp [21], see Theorem 5.1, for when the unit circle is covered by random arcs. To be able to use Shepp's condition, the  $\mu$ -measure of a certain set of trajectories must be calculated. This is done in the key lemma of the section, Lemma 5.2, which we think might be of independent interest. Lemma 5.2 has a somewhat surprising consequence, see Equation (76).

We now give some historical remarks concerning the study of visibility in various models. The problem of visibility was first studied by G.Pólya in [17] where he considered the visibility for a person at the origin and discs of radius  $R > 0$ , placed on the lattice  $\mathbb{Z}^2$ . For the Poisson Boolean model of continuum percolation in the Euclidean plane, an explicit expression is known for the probability that the visibility is larger than  $r$ , see Proposition 2.1 on p.4 in [3] (which uses a formula from [23]). Visibility in non-Euclidean spaces has been considered by R.Lyons in [15], where he studied the visibility on manifolds with negative curvature, see also Kahanes earlier works [8] [9] in the two-dimensional case. In the hyperbolic plane, visibility in so-called well behaved random sets was studied in [1] by Benjamini et. al.

The rest of the paper is organized as follows. In Section 2 we give the definitions of Brownian interacements and Brownian excursions, and give the precise formulations of our results. Section 3 contains some preliminary results needed for the proof of our main result for Brownian interacements. In Section 4 we prove the main result for Brownian interacements. The final section of the paper, Section 5, contains the proof of our main result for the Brownian excursion process.

We now introduce some notation. We denote by  $1\{A\}$  the indicator function of a set  $A$ . By  $A \Subset X$  we mean that  $A$  is a compact subset of a topological space  $X$ . Let  $a \in [0, \infty]$  and  $f, g$  be two functions. If  $\limsup_{x \rightarrow a} f/g = 0$  we write  $f = o(g(x))$  as  $x \rightarrow a$ , and if  $\limsup_{x \rightarrow a} f/g < \infty$  we write  $f = O(g(x))$  as  $x \rightarrow a$ . We write  $f(x) \sim g(x)$  as  $x \rightarrow a$  to indicate that  $\lim_{x \rightarrow a} (f(x)/g(x)) = 1$  and  $f(x) \lesssim g(x)$  as  $x \rightarrow a$  to indicate that  $f(x) \leq g(x)(1 + o(1))$  as  $x \rightarrow a$ . For  $x \in \mathbb{R}^d$  and  $r > 0$ , let  $B(x, r) = \{y : |x - y| \leq r\}$  and  $B(r) = B(0, r)$ . For  $A \subset \mathbb{R}^d$  define

$$A^t := \left\{ x \in \mathbb{R}^d : \text{dist}(x, A) \leq t \right\},$$

to be the closed  $t$ -neighbourhood of  $A$ . For  $x, y \in \mathbb{R}^d$  let  $[x, y]$  be the (straight) line segment between  $x$  and  $y$ .

Finally, we describe the notation and the convention for constants used in this paper. We will let  $c, c', c''$  denote positive finite constants that are allowed to depend on the dimension  $d$  and the thickness  $\rho$  only, and their values might change from place to place, even on the same line. With numbered constants  $c_i$ ,  $i \geq 1$ , we denote constants that are defined where they first appear within a proof, and stay the same for the rest of the proof. If a constant depends on another parameter, for example the intensity of the underlying Poisson point process, this is indicated.

## 2 Preliminaries

### 2.1 Brownian interlacements

We begin with the setup as in [25]. Let  $C = C(\mathbb{R}; \mathbb{R}^d)$  denote the continuous functions from  $\mathbb{R}$  to  $\mathbb{R}^d$  and let  $C_+ = C(\mathbb{R}_+; \mathbb{R}^d)$  denote the continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}^d$ . Define

$$W = \{x \in C : \lim_{|t| \rightarrow \infty} |x(t)| = \infty\} \text{ and } W_+ = \{x \in C_+ : \lim_{t \rightarrow \infty} |x(t)| = \infty\}.$$

On  $W$  we let  $X_t$ ,  $t \in \mathbb{R}$ , denote the canonical process, i.e.  $X_t(w) = w(t)$  for  $w \in C$ , and let  $\mathcal{W}$  denote the  $\sigma$ -algebra generated by the canonical processes. Moreover we let  $\theta_x$ ,  $x \in \mathbb{R}$  denote the shift operators acting on  $\mathbb{R}$ , that is  $\theta_x : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto y + x$ . We extend this notion to act on  $C$  by composition as

$$\theta_x : C \rightarrow C, f \mapsto f \circ \theta_x.$$

Similarly, on  $W_+$ , we define the canonical process  $X_t$ ,  $t \geq 0$ , the shifts  $\theta_h$ ,  $h \geq 0$ , and the sigma algebra  $\mathcal{W}_+$  generated by the canonical processes. We define the following random times corresponding to the canonical processes. For  $F \subset \mathbb{R}^d$  closed and  $w \in W_+$ , the **entrance time** is defined as  $H_F(w) = \inf\{t \geq 0 : X_t(w) \in F\}$  and the **hitting time** is defined as  $\tilde{H}_F(w) = \inf\{t > 0 : X_t(w) \in F\}$ . For  $K \Subset \mathbb{R}^d$  the **time of last visit** to  $K$  for  $w \in W_+$  is defined as  $L_K(w) = \sup\{t > 0 : X_t(w) \in K\}$ . The entrance time for  $w \in W$  is defined similarly, but  $t > 0$  is replaced by  $t \in \mathbb{R}$ . On  $W$ , we introduce the equivalence relation  $w \sim w' \Leftrightarrow \exists h \in \mathbb{R} : \theta_h w = w'$  and we denote the quotient space by  $W^* = W / \sim$  and let

$$\pi : W \rightarrow W^*, w \mapsto w^*,$$

denote the canonical projection. Moreover, we let  $\mathcal{W}^*$  denote the largest  $\sigma$ -algebra such that  $\pi$  is a measurable function, i.e.  $\mathcal{W}^* = \{\pi^{-1}(A) : A \in \mathcal{W}\}$ . We denote  $W_K \subset W$  all trajectories which enter  $K$ , and  $W_K^*$  the associated projection. We let  $P_x$  be the Wiener measure on  $C$  with the canonical process starting at  $x$ , and we denote  $P_x^B(\cdot) = P_x(\cdot | H_B = \infty)$  the probability measure conditioned on the event that the Brownian motion never hits  $B$ . For a finite measure  $\lambda$  on  $\mathbb{R}^d$  we define

$$P_\lambda = \int P_x \lambda(dx).$$

The transition density for the Brownian motion on  $\mathbb{R}^d$  is given by

$$p(t, x, y) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x - y|^2}{2t}\right) \quad (1)$$

and the Greens function is given by

$$G(x, y) = G(x - y) := \int_0^\infty p(t, x, y) dt = c_d |x - y|^{d-2},$$

where  $c_d$  is some dimension dependent constant, see Theorem 3.33 p.80 in [16].

Following [25] we introduce the following potential theoretic framework. For  $K \Subset \mathbb{R}^d$  let  $\mathcal{P}(K)$  be the space of probability measures supported on  $K$  and introduce the energy functional

$$E_K(\lambda) = \int_{K \times K} G(x,y) \lambda(dx) \lambda(dy), \quad \lambda \in \mathcal{P}(K). \quad (2)$$

The Newtonian capacity of  $K \Subset \mathbb{R}^d$  is defined as

$$\text{cap}(K) := \left( \inf_{\lambda \in \mathcal{P}(K)} \{E_K(\lambda)\} \right)^{-1}, \quad (3)$$

see for instance [2], [18] or [16]. It is the case that

$$\text{the capacity is a strongly sub-additive and monotone set-function.} \quad (4)$$

Let  $e_K(dy)$  be the equilibrium measure, which is the finite measure that is uniquely determined by the last exit formula, see Theorem 8.8 in [16],

$$P_x(X(L_K) \in A) = \int_A G(x,y) e_K(dy), \quad (5)$$

and let  $\tilde{e}_K$  be the normalized equilibrium measure. By Theorem 8.27 on p. 240 in [16] we have that  $\tilde{e}_K$  is the unique minimizer of (2) and

$$\text{cap}(K) = e_K(K). \quad (6)$$

Moreover the support satisfies  $\text{supp } e_K(dy) = \partial K$ .

If  $B$  is a closed ball, we define the measure  $Q_B$  on  $W_B^0 := \{w \in W : H_B(w) = 0\}$  as follows:

$$Q_B [(X_{-t})_{t \geq 0} \subset A', X_0 \in dy, (X_t)_{t \geq 0} \subset A] := P_y^B(A') P_y(A) e_B(dy), \quad (7)$$

where  $A, A' \in \mathcal{W}_+$ . If  $K$  is compact, then  $Q_K$  is defined as

$$Q_K = \theta_{H_K} \circ (1\{H_K < \infty\} Q_B), \text{ for any closed ball } B \supseteq K.$$

As pointed out in [25] this definition is independent of the choice of  $B \supseteq K$  and coincides with (7) when  $K$  is a closed ball. We point out that Equation 2.21 of [25] says that

$$Q_K[(X_t)_{t \geq 0} \in \cdot] = P_{e_K}(\cdot). \quad (8)$$

From [25] we have the following theorem, which is Theorem 2.2 on p.564.

**Theorem 2.1.** *There exists a unique  $\sigma$ -finite measure  $\nu$  on  $(W^*, \mathcal{W}^*)$  such that for all  $K$  compact,*

$$\nu(\cdot \cap W_K^*) = \pi \circ Q_K(\cdot) \quad (9)$$

Now we introduce the space of point measures or configurations, where  $\delta$  is the usual Dirac measure:

$$\Omega = \left\{ \omega = \sum_{i \geq 0} \delta(w_i^*, \alpha_i) : (w_i^*, \alpha_i) \in W^* \times [0, \infty), \omega(W_K^* \times [0, \alpha]) < \infty, \forall K \Subset \mathbb{R}^d, \alpha \geq 0 \right\}, \quad (10)$$

and we endow  $\Omega$  with the  $\sigma$ -algebra  $\mathcal{M}$  generated by the evaluation maps

$$\omega \mapsto \omega(B), B \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+).$$

Furthermore, we let  $\mathbb{P}$  denote the law of the Poisson point process of  $W^* \times \mathbb{R}_+$  with intensity measure  $\nu \otimes d\alpha$ . The Brownian interlacement is then defined as the random closed set

$$\text{BI}_\alpha^\rho(\omega) := \bigcup_{\alpha_i \leq \alpha} \bigcup_{s \in \mathbb{R}} B(w_i(s), \rho), \quad (11)$$

where  $\omega = \sum_{i \geq 0} \delta(w_i^*, \alpha_i) \in \Omega$  and  $\pi(w_i) = w_i^*$ . We then let  $\mathcal{V}_{\alpha, \rho} = \mathbb{R}^d \setminus \text{BI}_\alpha^\rho$  denote the vacant set.

The law of  $\text{BI}_\alpha^\rho$  is characterized as follows. Let  $\Sigma$  denote the family of all closed sets of  $\mathbb{R}^d$  and let  $\mathbb{F} := \sigma(F \in \Sigma : F \cap K = \emptyset, K \text{ compact})$ . The law of the interlacement set,  $\mathcal{Q}_\alpha^\rho$ , is a probability measure on  $(\Sigma, \mathbb{F})$  given by the following identity:

$$\mathcal{Q}_\alpha^\rho(\{F \in \Sigma : F \cap K = \emptyset\}) = \mathbb{P}(\text{BI}_\alpha^\rho \cap K = \emptyset) = e^{-\alpha \text{cap}(K^\rho)}. \quad (12)$$

For convenience, we also introduce the following notation. For  $\alpha > 0$  and  $\omega = \sum_{i \geq 1} \delta(w_i, \alpha_i) \in \Omega$ , we write

$$\omega_\alpha := \sum_{i \geq 1} \delta(w_i, \alpha_i) 1\{\alpha_i \leq \alpha\}. \quad (13)$$

Observe that under  $\mathbb{P}$ ,  $\omega_\alpha$  is a Poisson point process on  $W^*$  with intensity measure  $\alpha\nu$ . Note that, by Remark 2.3. (2) and Proposition 2.4 in [25] both  $\nu$  and  $\mathbb{P}$  are invariant under translations as well as linear isometries.

**Remark.** To get a better intuition of how this model works it might be good to think of the local structure of the random set  $\text{BI}_\alpha^\rho$ . This can be done in the following way, which uses (8). Let  $K \subset \mathbb{R}^d$  be a compact set. Let  $N_K \sim \text{Poisson}(\alpha \text{cap}(K))$ . Conditioned on  $N_K$ , let  $(y_i)_{i=1}^{N_K}$  be i.i.d. with distribution  $\tilde{e}_K$ . Conditioned on  $N_K$  and  $(y_i)_{i=1}^{N_K}$  let  $((B_i(t))_{t \geq 0})_{i=1}^{N_K}$  be a collection of independent Brownian motions in  $\mathbb{R}^d$  with  $B_i(0) = y_i$  for  $i = 1, \dots, N_K$ . We have the following distributional equality:

$$K \cap \text{BI}_\alpha^\rho \stackrel{d}{=} \left( \bigcup_{i=1}^{N_K} [B_i]^\rho \right) \cap K, \quad (14)$$

where  $[B_i]$  stands for the trace of  $B_i$ .

## 2.2 Results for the Brownian interlacements model in Euclidean space

The following theorem is our main result concerning visibility inside the vacant set of Brownian interlacements in  $\mathbb{R}^d$ .

**Theorem 2.2.** *There exist constants  $0 < c < c' < \infty$  depending only on  $d$ ,  $\rho$  and  $\alpha$  such that*

$$P_{\text{vis}}(r) \lesssim c' r^{2(d-1)} f(r), \quad d \geq 3, \quad (15)$$

$$P_{\text{vis}}(r) \gtrsim c r^{d-1} f(r), \quad d \geq 4, \quad (16)$$

as  $r \rightarrow \infty$ .

We believe that the lower bound in (16) is closer to the true asymptotic behaviour of  $P_{\text{vis}}(r)$  as  $r \rightarrow \infty$  than the upper bound in (15). Indeed, if for  $r > 0$  we let  $Z_r$  denote the set of points  $x \in \partial B(0, r)$  such that  $[0, x] \subset \mathcal{V}_{\alpha, \rho}$ , then the expected value of  $|Z_r|$  is proportional to  $r^{d-1} f(r)$ . We also observe that a consequence of Theorem 2.2 we obtain that  $P_{\text{vis}}(r) \rightarrow 0$  as  $r \rightarrow \infty$ . However, this fact can be obtained in simpler ways than Theorem 2.2.

## 2.3 Brownian excursions in the unit disc

The Brownian excursion measure on a domain  $S$  is a  $\sigma$ -finite measure on Brownian paths which is supported on the set of continuous paths,  $w = (w(t))_{0 \leq t \leq T_w}$ , that start and end on the boundary  $\partial S$  such that  $w(t) \in S, \forall t \in (0, T_w)$ . Its definition is found in for example [12], [29], see also [11], [13] for useful reviews. We now recall the definition and properties of the Brownian excursion measure in the case when  $S$  is the open unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ .

Let

$$W_{\mathbb{D}} := \{w \in C([0, T_w], \bar{\mathbb{D}}) : w(0), w(T_w) \in \partial \mathbb{D}, w(t) \in \mathbb{D}, \forall t \in (0, T_w)\}$$

and let  $X_t(w) = w(t)$  be the canonical process on  $W_{\mathbb{D}}$ . Let  $\mathcal{W}_{\mathbb{D}}$  be the sigma-algebra generated by the canonical processes. Moreover, for  $K \subset \mathbb{D}$  we let  $W_{K, \mathbb{D}}$  be the set of trajectories in  $W_{\mathbb{D}}$  that hit  $K$ . Let

$$\Omega_{\mathbb{D}} = \left\{ \omega = \sum_{i \geq 0} \delta(w_i, \alpha_i) : (w_i, \alpha_i) \in W_{\mathbb{D}} \times [0, \infty), \omega(W_{K, \mathbb{D}} \times [0, \alpha]) < \infty, \forall K \Subset \mathbb{D}, \alpha \geq 0 \right\}. \quad (17)$$

We endow  $\Omega_{\mathbb{D}}$  with the  $\sigma$ -algebra  $\mathcal{M}_{\mathbb{D}}$  generated by the evaluation maps

$$\omega \mapsto \omega(B), B \in \mathcal{W}_{\mathbb{D}} \otimes \mathcal{B}(\mathbb{R}_+).$$

For a probability measure  $\sigma$  on  $\mathbb{D}$ , denote by  $P_{\sigma}$  the law of Brownian motion with starting point chosen at random according to  $\sigma$ , stopped upon hitting  $\partial \mathbb{D}$ . (Note that  $P_{\sigma}$  has a different meaning if it occurs in a section concerning Brownian interlacements.)

For  $r > 0$ , let  $\sigma_r$  be the uniform probability measure on  $\partial B(0, r) \subset \mathbb{R}^2$ . The Brownian excursion measure on  $\mathbb{D}$  is defined as the limit

$$\mu = \lim_{\epsilon \rightarrow 0} \frac{2\pi}{\epsilon} P_{\sigma_{1-\epsilon}}. \quad (18)$$

See for example Chapter 5 in [11] for details. The measure  $\mu$  is a sigma-finite measure on  $W_{\mathbb{D}}$  with infinite mass.

As in [30] we can then define the Brownian excursion process as a Poisson point process on  $W_{\mathbb{D}} \times \mathbb{R}_+$  with intensity measure  $\mu \otimes d\alpha$  and we let  $\mathbb{P}_{\mathbb{D}}$  denote the probability measure corresponding to this process.

For  $\alpha > 0$ , the Brownian excursion set at level  $\alpha$  is then defined as

$$\text{BE}_{\alpha}(\omega) := \bigcup_{\alpha_i \leq \alpha} \bigcup_{s \geq 0} w_i(s), \quad \omega = \sum_{i \geq 0} \delta(w_i, \alpha_i) \in \Omega_{\mathbb{D}}, \quad (19)$$

and we let  $\mathcal{V}_{\alpha} = \mathbb{D} \setminus \text{BE}_{\alpha}$  denote the vacant set.

Proposition 5.8 in [11] says that  $\mu$ , and consequently  $\mathbb{P}_{\mathbb{D}}$ , are invariant under conformal automorphisms of  $\mathbb{D}$ . The conformal automorphisms of  $\mathbb{D}$  are given by

$$T_{\lambda, a} = \lambda \frac{z - a}{\bar{a}z - 1}, \quad |\lambda| = 1, \quad |a| < 1. \quad (20)$$

On  $\mathbb{D}$  we consider the hyperbolic metric  $\rho$  given by

$$\rho(u, v) = 2 \tanh^{-1} \left| \frac{u - v}{1 - \bar{u}v} \right| \quad \text{for } u, v \in \mathbb{D}.$$

We refer to  $\mathbb{D}$  equipped with  $\rho$  as the Poincaré disc model of 2-dimensional hyperbolic space  $\mathbb{H}^2$ . The metric  $\rho$  is invariant under  $(T_{\lambda, a})_{|\lambda|=1, |a|<1}$ .

The Brownian excursion process can in some sense be thought of as the  $\mathbb{H}^2$  analogue of the Brownian interlacements process due to the following reasons. As already mentioned that the law of the Brownian excursion process is invariant under the conformal automorphisms of  $\mathbb{D}$ , which are isometries of  $\mathbb{H}^2$ . Moreover, Brownian motion in  $\mathbb{H}^2$  started at  $x \in \mathbb{D}$  can be seen as a time-changed Brownian motion started at  $x$  stopped upon hitting  $\partial\mathbb{D}$ , see Example 3.3.3 on p.84 in [7]. In addition, we can easily calculate the  $\mu$ -measure of trajectories that hit a ball as follows. First observe that for  $r < 1$

$$\begin{aligned} \mu(\{\gamma : \gamma \cap B(0, r) \neq \emptyset\}) &= \lim_{\epsilon \rightarrow 0} 2\pi \epsilon^{-1} P_{\sigma_{1-\epsilon}}(H_{B(0, r)} < \infty) \\ &= \lim_{\epsilon \rightarrow 0} \frac{2\pi \log(1 - \epsilon)}{\epsilon \log(r)} = -\frac{2\pi}{\log(r)}, \end{aligned} \quad (21)$$

where we used Theorem 3.18 of [16] in the penultimate equality. For  $r_h \geq 0$  let  $B_{\mathbb{H}^2}(x, r_h) = \{y \in \mathbb{D} : \rho(x, y) \leq r_h\}$  be the closed hyperbolic ball centered at  $x$  with hyperbolic radius  $r_h$ . Then  $B_{\mathbb{H}^2}(0, r_h) = B(0, (e^{r_h} - 1)/(e^{r_h} + 1))$  so that

$$\mu(\{\gamma : \gamma \cap B_{\mathbb{H}^2}(0, r_h) \neq \emptyset\}) = -\frac{2\pi}{\log\left(\frac{e^{r_h} - 1}{e^{r_h} + 1}\right)} = \frac{2\pi}{\log(\coth(r_h/2))}.$$

The last expression can be recognized as the hyperbolic capacity (see [6] for definition) of a hyperbolic ball of radius  $r_h$ , since according to Equation 4.23 in [6]

$$\text{cap}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(0, r_h)) = \left( \int_{r_h}^{\infty} \frac{1}{S(t)} dt \right)^{-1}, \quad (22)$$

where  $S(r_h) = 2\pi \sinh(r_h)$  is the circumference of a ball of radius  $r_h$  in the hyperbolic metric. The integral equals

$$\int_{r_h}^{\infty} \frac{1}{2\pi \sinh(t)} dt = \frac{1}{2\pi} [\log(\tanh(t/2))]_{r_h}^{\infty} = \frac{\log(\coth(r_h/2))}{2\pi},$$

which yields the expression

$$\text{cap}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(0, r_h)) = \frac{2\pi}{\log[\coth(r_h/2)]},$$

which coincides with (21).

We now define the event of interest in this section. Let

$$V_{\infty}^{\alpha} = \left\{ \theta \in [0, 2\pi) : [0, e^{i\theta}] \subset \mathcal{V}_{\alpha} \neq \emptyset \right\}. \quad (23)$$

If  $V_{\infty}^{\alpha}$  occurs, we say that we have visibility to infinity in the vacant set (since  $[0, e^{i\theta}]$  has infinite length in the hyperbolic metric). As remarked above, such a phenomena cannot occur for the Brownian interlacements model on  $\mathbb{R}^d$  ( $d \geq 3$ ).

## 2.4 Results for the Brownian excursions process

Our main result (Theorem 2.3) for the Brownian excursion process is that we have a phase transition for visibility to infinity in the vacant set. We also determine the critical level for this transition and what happens at the critical level.

**Theorem 2.3.** *It is the case that*

$$\begin{aligned} \mathbb{P}_{\mathbb{D}}(V_{\infty}^{\alpha}) &> 0, & \alpha < \pi/4, \\ \mathbb{P}_{\mathbb{D}}(V_{\infty}^{\alpha}) &= 0, & \alpha \geq \pi/4. \end{aligned} \quad (24)$$

**Remark.** A similar phase-transition for visibility to infinity was proven to hold for so called well-behaved random sets in the hyperbolic plane in [1]. One example of a well-behaved random set is the vacant set of the Poisson-Boolean model of continuum percolation with balls of deterministic radii. In this model, balls of some fixed radius are centered around the points of a homogeneous Poisson point process in  $\mathbb{H}^2$ , and the vacant set is the complement of the union of those balls. In this case, a phase-transition for visibility was known to hold earlier, see [15].

**Remark.** It is easy to see that

$$\mathbb{P}_{\mathbb{D}}([0, e^{i\theta}] \subset \mathcal{V}_{\alpha}) = 0 \text{ for every } \theta \in [0, 2\pi) \text{ and every } \alpha > 0. \quad (25)$$

Hence, the set  $\{\theta \in [0, 2\pi) : [0, e^{i\theta}) \subset \mathcal{V}_\alpha\}$  has Lebesgue measure 0 a.s. when  $\alpha > 0$ . It could be of interest to determine the Hausdorff dimension of  $\{\theta \in [0, 2\pi) : [0, e^{i\theta}) \subset \mathcal{V}_\alpha\}$  on the event that this set is non-empty. This was for example done for well-behaved random sets in the hyperbolic plane in [26].

### 3 Preliminary results for the Euclidean case

In this section we collect some preliminary results needed for the proof of Theorem 2.2. The parameters  $\alpha > 0$  and  $\rho > 0$  will be kept fixed, so for brevity we write  $\mathcal{V}$  and  $\mathbf{BI}$  for  $\mathcal{V}_{\alpha, \rho}$  and  $\mathbf{BI}_\alpha^\rho$  respectively. We now introduce some additional notation. For  $A, B \in \mathbb{R}^d$  define the event

$$A \stackrel{g}{\leftrightarrow} B := \{\exists x \in A, y \in B : [x, y] \subset \mathcal{V}\}. \quad (26)$$

Then

$$P_{\text{vis}}(r) = \mathbb{P}\left(0 \stackrel{g}{\leftrightarrow} \partial B(r)\right), \quad (27)$$

$$f(r) = \mathbb{P}\left(0 \stackrel{g}{\leftrightarrow} xr\right), x \in S^{d-1}, \quad (28)$$

where  $S^{d-1} = \partial B(1)$ . For  $L, \rho > 0$  let

$$[0, L]_\rho := \left\{x = (x_1, x') \in \mathbb{R}^d : x_1 \in [0, L], |x'| \leq \rho\right\}. \quad (29)$$

For  $x, y \in \mathbb{R}^d$  let  $[x, y]_\rho = R_{x, y}([0, |x - y|]_\rho)$  where  $R_{x, y}$  is an isometry on  $\mathbb{R}^d$  mapping 0 to  $x$  and  $(|x - y|, 0, \dots, 0)$  to  $y$ . In other words,  $[x, y]_\rho$  is the finite cylinder with base radius  $\rho$  and with central axis running between  $x$  and  $y$ . Using estimates of the capacity of  $[0, L]_1$  from [18] we easily obtain estimates of the capacity of  $[0, L]_\rho$  for general  $\rho$  as follows.

**Lemma 3.1.** *For every  $L_0 \in (0, \infty)$  and  $\rho_0 \in (0, \infty)$  there are constants  $c, c' \in (0, \infty)$  (depending on  $L_0, \rho_0$  and  $d$ ) such that for  $L \geq L_0$ ,  $\rho \leq \rho_0$ ,*

$$\begin{aligned} c\rho^{d-3}L &\leq \text{cap}([0, L]_\rho) \leq c'\rho^{d-3}L, \quad d > 3, \\ cL/(\log(L/\rho)) &\leq \text{cap}([0, L]_\rho) \leq c'L/(\log(L/\rho)), \quad d = 3. \end{aligned}$$

*Proof.* Fix  $L_0, \rho_0 \in (0, \infty)$  and consider  $L \geq L_0$  and  $\rho \leq \rho_0$ . Note that  $[0, L]_\rho = \rho[0, L/\rho]_1$ . Hence by the homogeneity property of the capacity, see Proposition 3.4 p.67 in [18], we have

$$\text{cap}([0, L]_\rho) = \rho^{d-2} \text{cap}([0, L/\rho]_1).$$

We then utilize the following bounds, see Proposition 1.12 p.60 and Proposition 3.4 p.67 in [18]: For each  $L'_0 \in (0, \infty)$  there are constants  $c, c'$  such that

$$\begin{aligned} cL &\leq \text{cap}([0, L]_1) \leq c'L, \quad d > 3, \\ cL/\log(L) &\leq \text{cap}([0, L]_1) \leq c'L/\log(L), \quad d = 3, \end{aligned}$$

for  $L \geq L'_0$ . The results follows, since  $L/\rho \geq L_0/\rho_0$ .  $\square$

Observe that by invariance, Proposition 3.4 p.67 in [18],

$$\text{cap}([x,y]_\rho) = \text{cap}([0,|x-y|]_\rho).$$

Next, we discuss the probability that a given line segment of length  $r$  is contained in  $\mathcal{V}$ , that is  $f(r)$ . Note that for  $x, y \in \mathbb{R}^d$ ,

$$\{x \stackrel{\rho}{\leftrightarrow} y\} = \left\{ \omega \in \Omega : \omega_\alpha \left( W_{[x,y]_\rho}^* \right) = 0 \right\}.$$

Since under  $\mathbb{P}$ ,  $\omega$  is a Poisson point process with intensity measure  $\nu \otimes d\alpha$  we get that

$$f(|x-y|) = e^{-\alpha \text{cap}([x,y]_\rho)}. \quad (30)$$

Since  $[x,y]_\rho$  is the union of the cylinder  $[x,y]_\rho$  and two half-spheres of radius  $\rho$ , it follows using (4) that

$$c(\alpha) e^{-\alpha \text{cap}([x,y]_\rho)} \leq f(|x-y|) \leq e^{-\alpha \text{cap}([x,y]_\rho)}. \quad (31)$$

The next lemma will be used in the proof of (16).

**Lemma 3.2.** *Let  $d \geq 4$  and  $L$  be a bi-infinite line. Let  $L_r$  be a line segment of length  $r \geq 1$ . There are constants  $c(d, \rho), c'(d, \rho)$  such that*

$$\nu(W_{L_r}^* \setminus W_{L\rho}^*) \geq (1 - c \text{dist}(L_r, L)^{-(d-3)}) \nu(W_{L_r}^*). \quad (32)$$

whenever  $\text{dist}(L_r, L) \geq c'$ .

*Proof.* For simplicity we assume through the proof that  $r \geq 1$  is an integer and that one of the endpoints of  $L_r$  minimizes the distance between  $L$  and  $L_r$ . The modification of the proof to the case of general  $r \geq 1$  and general orientations of the line and the line-segment is straightforward. We write

$$\nu(W_{L_r}^*) = \nu(W_{L_r}^* \setminus W_{L\rho}^*) + \nu(W_{L_r}^* \cap W_{L\rho}^*), \quad (33)$$

and focus on finding a useful upper bound of the second term of the right hand side.

We now write  $L = (\gamma_1(t))_{t \in \mathbb{R}}$ , where  $\gamma_1$  is parametrized to be unit speed and such that  $\text{dist}(L_r, \gamma_1(0)) = \text{dist}(L_r, L)$ . Similarly, we write  $L_r = (\gamma_2(t))_{0 \leq t \leq r}$  where  $\gamma_2$  has unit speed and  $\text{dist}(\gamma_2(0), L) = \text{dist}(L_r, L)$ . For  $i \in \mathbb{Z}$  and  $0 \leq j \leq r-1$  let  $y_i = \gamma_1(i)$  and let  $z_j = \gamma_2(j)$ . Choose  $s = s(\rho) < \infty$  such that

$$L^\rho \subset \bigcup_{i \in \mathbb{Z}} B(y_i, s) \text{ and } L_r^\rho \subset \bigcup_{i=0}^{r-1} B(z_i, s).$$

We now have that

$$\begin{aligned} \nu(W_{L_r}^* \cap W_{L\rho}^*) &\leq \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \nu(W_{B(z_j, s)}^* \cap W_{B(y_i, s)}^*) \\ &\leq \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \frac{c}{|z_j - y_i|^{(d-2)}} \leq cr \sum_{i \in \mathbb{Z}} \frac{1}{|z_0 - y_i|^{d-2}} \\ &\leq cr \sum_{i \in \mathbb{Z}} \frac{1}{(\text{dist}(L, L_r)^2 + i^2)^{\frac{d-2}{2}}} \leq c_1 r \text{dist}(L, L_r)^{-(d-3)}, \end{aligned}$$

where the second inequality follows from Lemma 2.1 on p.14 in [14]. Combining this with the fact from Lemma 3.1 that  $\nu(W_{L_r^\rho}^*) \geq c_2 r$  whenever  $r \geq 1$ , we get that

$$\nu(W_{L_r^\rho}^* \cap W_{L_r}^*) \leq \frac{c_1}{c_2} \nu(W_{L_r^\rho}^*) \text{dist}(L, L_r)^{-(d-3)},$$

which together with (33) gives the result.  $\square$

**Remark.** Observe the the Lemma above implies that for every  $r > 1$ , and every line  $L$  and line-segment  $L_r$  of length  $r$  satisfying  $\text{dist}(L, L_r) > c$ , we have

$$\nu(W_{L_r^\rho}^* \setminus W_{L_r}^*) \geq \frac{1}{2} \nu(W_{L_r^\rho}^*).$$

It is easy to generalize the statement to hold for every  $r > 0$ .

## 4 Proof of Theorem 2.2

We split the proof of Theorem 2.2 into the proofs of two propositions, Proposition 4.1 which is the lower bound (16) and Proposition 4.2 which is the upper bound (15).

### 4.1 The lower bound

To get a lower bound we will utilize the second moment method. More precisely we shall modify the arguments from the proof of Lemma 3.6 on p.332 in [1]. Let  $\sigma(dx)$  denote the surface measure of  $S^{d-1}$ , and for  $r > 0$  define

$$Y_r := \left\{ x \in S^{d-1} : [0, rx] \subset \mathcal{V} \right\}, \quad (34)$$

$$y_r := |Y_r| = \int_{S^{d-1}} 1_{Y_r}(x) \sigma(dx). \quad (35)$$

The expectation and the second moment of  $y_r$  are computed using Fubini's theorem:

$$\mathbb{E}(y_r) = |S^{d-1}| f(r) \quad (36)$$

$$\mathbb{E}(y_r^2) = \int_{(S^{d-1})^2} \mathbb{P}(x, x' \in Y_r) \sigma(dx) \sigma(dx'), \quad (37)$$

where  $f(r)$  is given by (30) above. The crucial part of the proof of the lower bound in (15) is estimating (37) from above.

**Proposition 4.1.** *Let  $d \geq 4$ . There exist constants  $c(\alpha), c'$  such that*

$$P_{vis}(r) \geq c r^{d-1} f(r) \text{ for all } r \geq c'. \quad (38)$$

*Proof.* For  $x \in S^{d-1}$  let  $L_\infty(x)$  be the infinite half-line starting in 0 and passing through  $x$ . For  $x, x' \in S^{d-1}$  define  $\theta = \theta(x, x') := \arccos(\langle x, x' \rangle)$  to be the angle between the two half-lines  $L_\infty(x)$  and  $L_\infty(x')$ . From Lemma 3.2 and the remark thereafter we know that there is a constant  $c_1$  such that for every  $r > 0$ , and every line  $L$  and line-segment  $L_r$  of length  $r$  satisfying  $\text{dist}(L, L_r) \geq c_1$ , we have

$$\nu(W_{L_r}^* \setminus W_L^*) \geq \frac{1}{2} \nu(W_{L_r}^*). \quad (39)$$

Now define  $g(\theta) \in (0, \infty)$  by the equation

$$\text{dist}(L_\infty(x), L_\infty(x') \setminus [0, g(\theta)x']) = c_1. \quad (40)$$

Elementary trigonometry shows that if  $\theta \in [0, \pi/2]$  we have

$$g(\theta) = \frac{c_1}{\sin(\theta)},$$

and for  $\theta \in [\pi/2, \pi]$  it is easy to see that we have  $g(\theta) \leq c$ . Now, for  $x, x' \in S^{d-1}$ ,

$$\begin{aligned} \mathbb{P}(x, x' \in Y_r) &\leq \mathbb{P}([0, rx] \subset \mathcal{V}, [0, rx'] \setminus [0, g(\theta)x'] \subset \mathcal{V}) \\ &= \mathbb{P}\left(\omega_\alpha\left(W_{[0, rx]}^*\right) = 0, \omega_\alpha\left(W_{([0, rx'] \setminus [0, g(\theta)x'])}^*\right) = 0\right) \\ &\leq \mathbb{P}\left(\omega_\alpha\left(W_{[0, rx]}^*\right) = 0, \omega_\alpha\left(W_{([0, rx'] \setminus [0, g(\theta)x'])}^* \setminus W_{[0, rx]}^*\right) = 0\right) \\ &\stackrel{\text{indep.}}{=} \mathbb{P}\left(\omega_\alpha\left(W_{[0, rx]}^*\right) = 0\right) \mathbb{P}\left(\omega_\alpha\left(W_{([0, rx'] \setminus [0, g(\theta)x'])}^* \setminus W_{[0, rx]}^*\right) = 0\right) \\ &= f(r) \exp\left\{-\alpha \nu\left(W_{([0, rx'] \setminus [0, g(\theta)x'])}^* \setminus W_{[0, rx]}^*\right)\right\} \\ &\stackrel{(39)}{\leq} f(r) \exp\left\{-\frac{\alpha}{2} \nu\left(W_{(0, ((r-g(\theta)) \vee 0)x]}^*\right)\right\} \leq f(r) e^{-(c_2(\alpha)(r-g(\theta)) \vee 0)} c(\alpha), \end{aligned}$$

where the last inequality follows from Lemma 3.1. Hence, in order to get an upper bound of (37) we want to get an upper bound of

$$I = \int_{(S^{d-1})^2} \exp\{-c_2((r - g(\theta)) \vee 0)\} \sigma(dx) \sigma(dx'). \quad (41)$$

In spherical coordinates  $\theta, \theta_1, \dots, \theta_{d-2}$ , we get, with  $A(\theta_1, \dots, \theta_{d-2}) = \{(\theta_1, \dots, \theta_{d-2}) : 0 \leq \theta_i < 2\pi \text{ for all } i\}$ ,

$$\begin{aligned} I &= \int_{\theta=0}^{\pi/2} \int_{A(\theta_1, \dots, \theta_{d-2})} \exp\left\{-c_2\left(r - \frac{c_1}{\sin(\theta)}\right) \vee 0\right\} \sin^{d-2}(\theta) \sin^{d-3}(\theta_1) \cdots \sin(\theta_{d-3}) d\theta d\theta_1 \cdots d\theta_{d-2} \\ &\quad + \int_{\theta=\pi/2}^{\pi} \int_{A(\theta_1, \dots, \theta_{d-2})} \exp\{-c_2((r - c) \vee 0)\} \sin^{d-2}(\theta) \sin^{d-3}(\theta_1) \cdots \sin(\theta_{d-3}) d\theta d\theta_1 \cdots d\theta_{d-2} \\ &= I_1 + I_2. \end{aligned}$$

We now find an upper bound on the integral  $I_1$ . We get that

$$\begin{aligned} I_1 &\leq c_3 \int_0^{\pi/2} \exp \left\{ -c_2 \left( \left( r - \frac{c_1}{\sin(\theta)} \right) \vee 0 \right) \right\} \sin^{d-2}(\theta) d\theta \\ &= c_3 \left( \int_0^{\arcsin c_1/r} \sin^{d-2}(\theta) d\theta + \int_{\arcsin c_1/r}^{\pi/2} e^{-c_2 \left( r - \frac{c_1}{\sin(\theta)} \right)} \sin^{d-2}(\theta) d\theta \right). \end{aligned} \quad (42)$$

For the first of the two integrals above we get

$$\int_0^{\arcsin c_1/r} \sin^{d-2}(\theta) d\theta \leq c \int_0^{c_1/r} \theta^{d-2} d\theta = c' r^{-(d-1)}. \quad (43)$$

For the second integral in (42) we get (using that  $1/\sin(\theta) - 1/\theta$  can be extended to a uniformly continuous function on  $[0, \pi/2]$ )

$$\begin{aligned} &\int_{\arcsin c_1/r}^{\pi/2} e^{-c_2 \left( r - \frac{c_1}{\sin(\theta)} \right)} \sin^{d-2}(\theta) d\theta \leq c e^{-c_2 r} \int_{c_1/r}^{\pi/2} e^{c_1 c_2 / \theta} \theta^{d-2} d\theta = \\ &= c e^{-c_2 r} \int_{2/\pi}^{r/c_1} e^{c_1 c_2 t} t^{-d} dt = c e^{-c_2 r} \int_{2c_1 c_2 / \pi}^{c_2 r} e^y y^{-d} dy \\ &= c e^{-c_2 r} \int_{2c_1 c_2 / \pi}^{c_2 r/2} e^y y^{-d} dy + c e^{-c_2 r} \int_{c_2 r/2}^{c_2 r} e^y y^{-d} dy \\ &\leq c e^{-c_2 r/2} \int_{2c_1 c_2 / \pi}^{c_2 r/2} y^{-d} dy + c \int_{c_2 r/2}^{c_2 r} y^{-d} dy \leq c r^{-(d-1)}. \end{aligned} \quad (44)$$

Moreover, it is easy to see that

$$I_2 = O(e^{-cr}). \quad (45)$$

Putting equations (37), (41), (42), (43), (44) and (45) together, we obtain that for all  $r$  large enough,

$$\mathbb{E}[y_r^2] \leq c f(r) r^{-(d-1)}. \quad (46)$$

From (36), (46) and the second moment method we get that for all  $r$  large enough

$$P_{\text{vis}}(r) \geq \frac{\mathbb{E}(y_r)^2}{\mathbb{E}(y_r^2)} \geq c r^{d-1} f(r),$$

finishing the proof of the proposition.  $\square$

## 4.2 The upper bound

The next proposition is (15) in Theorem 2.2.

**Proposition 4.2.** *There exists a constant  $c < \infty$  depending only on  $d$ ,  $\rho$  and  $\alpha$  such that*

$$P_{\text{vis}}(r) \lesssim c r^{2(d-1)} f(r), \quad d \geq 3. \quad (47)$$

*Proof.* Fix  $r > 0$ ,  $x, y \in \mathbb{R}^d$  and  $\epsilon \in (0, \rho)$ . Let  $M(x, y, \epsilon) = \omega_\alpha(W_{[x, y]^{\rho - \epsilon}}^*)$  and let  $A(x, y, \epsilon)$  be the event that there is a connected component of  $[x, y]^\epsilon \cap \mathcal{V}$  that intersects both  $B(x, \epsilon)$  and  $B(y, \epsilon)$ . Observe that on the event that  $M(x, y, \epsilon) \geq 1$ , there is some  $z \in [x, y]$  such that  $d(z, \text{Bl}) \leq \rho - \epsilon$ . For this  $z$ , we have  $B(z, \epsilon) \subset \text{Bl}$ . Any continuous curve  $\gamma \subset [x, y]^\epsilon$  intersecting both  $B(x, \epsilon)$  and  $B(y, \epsilon)$  must also intersect  $B(z, \epsilon)$ . Hence,  $\{M(x, y, \epsilon) \geq 1\} \subset A(x, y, \epsilon)^c$ , and we get that

$$A(x, y, \epsilon) \subset \{M(x, y, \epsilon) = 0\}. \quad (48)$$

Now we let

$$N(\epsilon, r) = \inf \left\{ k \in \mathbb{N} : \exists x_1, x_2, \dots, x_k \in \partial B(r) \text{ such that } \bigcup_{i=1}^k B(x_i, \epsilon) \supset \partial B(r) \right\} \quad (49)$$

be the covering number for a sphere of radius  $r$ , and note that  $N(\epsilon, r) = O((r/\epsilon)^{d-1})$ . For each  $r > 0$ , let  $(x_i)_{i=1}^{N(\epsilon, r)}$  be a set of points on  $\partial B(r)$  such that  $\partial B(r) \subset \bigcup_{i=1}^{N(\epsilon, r)} B(x_i, \epsilon)$ . If  $\{0 \stackrel{g}{\leftrightarrow} \partial B(r)\}$  occurs there exists a  $j \in \{1, 2, \dots, N(\epsilon, r)\}$  such that  $A(0, x_j, \epsilon)$  occurs. Hence, by the union bound and rotational invariance (Equation 2.28 in [25]),

$$\begin{aligned} P_{\text{vis}}(r) &\leq P \left( \bigcup_{i=1}^{N(\epsilon, r)} A(0, x_i, \epsilon) \right) \leq N(\epsilon, r) P(A(0, x_1, \epsilon)) \\ &\stackrel{(48)}{\leq} O((r/\epsilon)^{d-1}) \mathbb{P}(M(0, x_1, \epsilon) = 0). \end{aligned} \quad (50)$$

Fix  $x \in S^{d-1}$  and let  $K_1 = K_1(r, \rho) = [0, rx]^\rho$  and  $K_2 = K_2(r, \rho, \epsilon) = [0, rx]^{\rho - \epsilon}$ . Then

$$f(r) = e^{-\text{acap}(K_1)}$$

and

$$\mathbb{P}(M(0, x_1, \epsilon) = 0) = e^{-\text{acap}(K_2)}.$$

Hence,

$$\mathbb{P}(M(0, x_1, \epsilon) = 0) = f(r) e^{\alpha(\text{cap}(K_1) - \text{cap}(K_2))} \quad (51)$$

We will now let  $\epsilon = \epsilon(r) = 1/r$  for  $r \geq \rho^{-1}$  and show that

$$\text{cap}(K_1) - \text{cap}(K_2) = O(1), \quad r \rightarrow \infty. \quad (52)$$

Let  $((B_i(t))_{t \geq 0})_{i \geq 1}$  be a collection of i.i.d. processes with distribution  $P_{\tilde{e}_{K_1}}$  where  $\tilde{e}_{K_1} = e_{K_1} / \text{cap}(K_1)$ . Recall that  $[B_i]$  stands for the trace of  $B_i$ . Using the local description of the Brownian interlacements, see Equation (14), we see that

$$\omega_\alpha(W_{K_1}^* \setminus W_{K_2}^*) \stackrel{d}{=} \sum_{i=1}^{N_{K_1}} 1\{[B_i] \cap K_2 = \emptyset\}, \quad (53)$$

where  $N_{K_1}$  is a Poisson random variable with mean  $\alpha \text{cap}(K_1)$  which is independent of the collection  $((B_i(t))_{t \geq 0})_{i \geq 1}$ , and the sum is interpreted as 0 in case  $N_{K_1} = 0$ . Taking expectations of both sides in (53) we obtain that

$$\begin{aligned} \alpha \nu(W_{K_1}^* \setminus W_{K_2}^*) &= E \left[ \sum_{i=1}^{N_{K_1}} 1\{[B_i] \cap K_2 = \emptyset\} \right] \\ &= E[N_{K_1}] P([B_1] \cap K_2 = \emptyset) = \alpha \text{cap}(K_1) P([B_1] \cap K_2 = \emptyset), \end{aligned} \quad (54)$$

where we used the independence between  $N_{K_1}$  and  $((B_i(t))_{t \geq 0})_{i \geq 1}$  and the fact the  $B_i$ -processes are identically distributed.

Since  $K_2 \subset K_1$ , it follows that

$$\nu(W_{K_1}^* \setminus W_{K_2}^*) = \text{cap}(K_1) - \text{cap}(K_2). \quad (55)$$

From (54) and (55) it follows that

$$\text{cap}(K_1) - \text{cap}(K_2) = \text{cap}(K_1) P([B_1] \cap K_2 = \emptyset). \quad (56)$$

Next, we find a useful upper bound on the last factor on the right hand side of (56). Recall that for  $t > 0$  and  $x \notin B(0, t)$ ,

$$P_x(\tilde{H}_{B(0, t)} < \infty) = (t/|x|)^{d-2}, \quad (57)$$

see for example Corollary 3.19 on p.72 in [16]. Now,

$$P([B_1] \cap K_2 = \emptyset) = P_{\tilde{e}_{K_1}}(\tilde{H}_{K_2} = \infty) = \int_{\partial K_1} P_y(\tilde{H}_{K_2} = \infty) \tilde{e}_{K_1}(dy). \quad (58)$$

For  $z \in \partial K_1$  let  $z'$  be the orthogonal projection of  $z$  onto the line segment  $[0, rx]$ . Since  $B(z', \rho - \epsilon) \subset K_2$  we have

$$\{\tilde{H}_{K_2} = \infty\} \subset \{\tilde{H}_{B(z', \rho - \epsilon)} = \infty\}. \quad (59)$$

We now get that

$$\begin{aligned} P([B_1] \cap K_2 = \emptyset) &\stackrel{(58), (59)}{\leq} \int_{\partial K_1} P_y(\tilde{H}_{B(y', \rho - \epsilon)} = \infty) \tilde{e}_{K_1}(dy) \\ &\stackrel{(57)}{=} 1 - \left(\frac{\rho - \epsilon}{\rho}\right)^{d-2} = 1 - (1 - \epsilon/\rho)^{d-2} = O(1/r), \end{aligned}$$

where we recall that we made the choice  $\epsilon = 1/r$  for  $r \geq \rho^{-1}$  above. Combining this with the fact that  $\text{cap}(K_1) = O(r)$  and (56) now gives (52). Equations (50) and (51) and (52) finally give that

$$P_{\text{vis}}(r) \leq O\left(r^{2(d-1)}\right) f(r)$$

as  $r \rightarrow \infty$ . This establishes the upper bound in (15) □

## 5 Visibility for Brownian excursions in the unit disk

In this section, we give the proof of Theorem 2.3. The method of proof we use here is an adaption of the method used in paper III of [27], which is an extended version of the paper [1]. We first recall a result of Shepp [21] concerning circle covering by random intervals. Given a decreasing sequence  $(l_n)_{n \geq 1}$  of strictly positive numbers, we let  $(I_n)_{n \geq 1}$  be a sequence of independent open random intervals, where  $I_n$  has length  $l_n$  and is centered at a point chosen uniformly at random on  $\partial\mathbb{D}/(2\pi)$  (we divide by  $2\pi$  since Shepp's result is formulated for a circle of circumference 1). Let  $E := \limsup_n I_n$  be the random subset of  $\partial\mathbb{D}$  which is covered by infinitely many intervals from the sequence  $(I_n)_{n \geq 1}$  and let  $F := E^c$ . If  $\sum_{n=1}^{\infty} l_n = \infty$  then  $F$  has measure 0 a.s. but one can still ask if  $F$  is empty or non-empty in this case. Shepp [21] proved that

**Theorem 5.1.**  $P(F = \emptyset) = 1$  if

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{l_1 + l_2 + \dots + l_n} = \infty, \quad (60)$$

and  $P(F = \emptyset) = 0$  if the above sum is finite.

Theorem 5.1 is formulated for open intervals, but the result holds the same if the intervals are taken to be closed or half-open, see the remark on p.340 of [21].

A special case of Theorem 5.1, which we will make use of below, is that if  $c > 0$  and  $l_n = c/n$  for  $n \geq 1$ , then (as is easily seen from (60))

$$P(F = \emptyset) = 1 \text{ if and only if } c \geq 1. \quad (61)$$

Before we explain how we use Theorem 5.1, we introduce some additional notation. If  $\gamma \subset \bar{\mathbb{D}}$  is a continuous curve, it generates a "shadow" on the boundary of the unit disc. The shadow is the arc of  $\partial\mathbb{D}$  which cannot be reached from the origin by moving along a straight line-segment without crossing  $\gamma$ . More precisely, we define the arc  $\mathcal{S}(\gamma) \subseteq \partial\mathbb{D}$  by

$$\mathcal{S}(\gamma) = \{e^{i\theta} : [0, e^{i\theta}] \cap \gamma \neq \emptyset\},$$

and let  $\Theta(\gamma) = \text{length}(\mathcal{S}(\gamma))$ , where length stands for arc-length on  $\partial\mathbb{D}$ .

We now explain how we use Theorem 5.1 to prove Theorem 2.3. First we need some additional notation. For  $\omega = \sum_{i \geq 1} \delta_{(w_i, \alpha_i)} \in \Omega_{\mathbb{D}}$  and  $\alpha > 0$  we write  $\omega_{\alpha} = \sum_{i \geq 1} \delta_{(w_i, \alpha_i)} 1_{\{\alpha_i \leq \alpha\}}$ . Then under  $\mathbb{P}_{\mathbb{D}}$ ,  $\omega_{\alpha}$  is a Poisson point process on  $W_{\mathbb{D}}$  with intensity measure  $\alpha\mu$ . Each  $(w_i, \alpha_i) \in \omega_{\alpha}$  generates a shadow  $\mathcal{S}(w_i) \subseteq \partial\mathbb{D}$  and a corresponding shadow-length  $\Theta(w_i) \in [0, 2\pi]$ . The process of shadow-lengths

$$\Xi_{\alpha} := \sum_{(w_i, \alpha_i) \in \text{supp}(\omega_{\alpha})} \delta_{\Theta(w_i)} 1_{\{\Theta(w_i) < 2\pi\}}$$

is a non-homogeneous Poisson point process on  $(0, 2\pi)$ , and we calculate the intensity measure of this Poisson point process below, see (76). Since Brownian motion started

inside  $\mathbb{D}$  stopped upon hitting  $\partial\mathbb{D}$  has a positive probability to make a full loop around the origin, there might be a random number of shadows that have length  $2\pi$  which we have thrown away in the definition of  $\Xi_\alpha$ . However, this number will be a Poisson random variable with finite mean (see the paragraph above (75)), so those shadows will not cause any major obstructions. Now, for  $i \geq 1$ , we denote by  $\Theta_{(i),\alpha}$  the length of the  $i$ :th longest shadow in  $\text{supp}(\Xi_\alpha)$ . We then show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{(\Theta_{(1),\alpha} + \Theta_{(2),\alpha} + \dots + \Theta_{(n),\alpha})/(2\pi)} = \infty \text{ a.s.} \quad (62)$$

if  $\alpha \geq \pi/4$  and finite a.s. otherwise, from which Theorem 2.3 easily will follow using Theorem 5.1.

We now recall some facts of one-dimensional Brownian motion which we will make use of. If  $(B(t))_{t \geq 0}$  is a one-dimensional Brownian motion, its range up to time  $t > 0$  is defined as

$$R(t) = \sup_{s \leq t} B(s) - \inf_{s \leq t} B(s).$$

The density function of  $R(t)$  is denoted by  $h(r,t)$  and we write  $h(r)$  for  $h(r,1)$ . An explicit expression of  $h(r,t)$  can be found in [5]. The expectation of  $R(t)$  is also calculated in [5]. In particular,

$$E[R(1)] = 2\sqrt{\frac{2}{\pi}}. \quad (63)$$

Let  $(B(t))_{t \geq 0}$  be a one-dimensional Brownian motion with  $B(0) = a \in \mathbb{R}$ . Let  $H_a = \inf\{t \geq 0 : B(t) = 0\}$  be the hitting time for the Brownian motion of the value 0. The density function of  $H_a$  is given by

$$f_a(t) = |a|e^{-a^2/2t}/\sqrt{2\pi t^3}, \quad t \geq 0. \quad (64)$$

Now let  $W = (W(t))_{t \geq 0}$  be a two-dimensional Brownian motion with  $W(0) = x \in \mathbb{D}$  stopped upon hitting  $\partial\mathbb{D}$ . Observe that the distribution of the length of the shadow generated by  $W$ ,  $\Theta(W)$ , depends on the starting point  $x$  only through  $|x|$ . The distribution of  $\Theta(W)$  might be known, but since we could not find any reference we include a derivation, which is found in Lemma 5.1 below. We thank K. Burdzy for providing a version of the arguments used in the proof of the lemma.

**Lemma 5.1.** *Suppose that  $W = (W(t))_{t \geq 0}$  is a Brownian motion started at  $x \in \mathbb{D} \setminus \{0\}$ , stopped upon hitting  $\partial\mathbb{D}$ . Then, for  $\theta \in (0, 2\pi]$ ,*

$$P(\theta \leq \Theta(W) \leq 2\pi) = \int_{\{(r,t) : r\sqrt{t} \geq \theta\}} f_{\log(|x|)}(t)h(r)dt dr. \quad (65)$$

*Proof.* Without loss of generality, suppose that the starting point  $x \in (0,1)$ . Consider the multi-valued function  $\phi(z) = \log(z)$ , which conformally maps  $\mathbb{D}$  onto the half-plane

$\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ . Note that for  $\theta \in [0, 2\pi)$ ,

$$\phi\left(\left\{re^{i\theta} : 0 < r < 1\right\}\right) = \{x + i(\theta + 2\pi k) : x < 0, k \in \mathbb{Z}\}. \quad (66)$$

By conformal invariance of Brownian motion, the law of the trace of  $\phi(W)$  is the same as the law of the trace of a standard two-dimensional Brownian  $\tilde{W} = (\tilde{W}(t))_{t \geq 0}$  started at  $\log(x) \in (-\infty, 0)$  and stopped upon hitting  $\{z \in \mathbb{C} : \operatorname{Re} z = 0\}$ . Let

$$T = \inf\left\{t > 0 : \operatorname{Re} \tilde{W}(t) = 0\right\} \quad \text{and} \quad R(T) = \sup_{s \leq T} \operatorname{Im} \tilde{W}(s) - \inf_{s \leq T} \operatorname{Im} \tilde{W}(s).$$

Using (66), we see that

$$P(\theta \leq \Theta \leq 2\pi) = P(\theta \leq R(T)). \quad (67)$$

Moreover,  $T$  and  $\operatorname{Im} \tilde{W}(t)$  are independent since  $T$  is determined by  $\operatorname{Re} \tilde{W}(t)$  and  $\operatorname{Re} \tilde{W}(t)$  and  $\operatorname{Im} \tilde{W}(t)$  are independent. Since  $T$  and  $\operatorname{Im} \tilde{W}(t)$  are independent we have by Brownian scaling

$$R(T) \stackrel{d}{=} \sqrt{T}R(1). \quad (68)$$

Hence

$$P(\theta \leq \Theta \leq 2\pi) \stackrel{(67), (68)}{=} P(\sqrt{T}R(1) \geq \theta) = \int_{r\sqrt{t} \geq \theta} f_{|\log(x)|}(t)h(r)dt dr, \quad (69)$$

finishing the proof of the lemma.  $\square$

In the next lemma, we calculate the intensity measure of  $\Xi_\alpha$ . For  $\theta \in (0, 2\pi]$  define

$$A_\theta = \{w \in W_{\mathbb{D}} : \theta \leq \mathcal{S}(w)\}. \quad (70)$$

**Lemma 5.2.** *For  $\theta \in (0, 2\pi]$*

$$\mu(A_\theta) = \frac{8}{\theta}. \quad (71)$$

*Proof.* By the definition of  $\mu$ , we must show that

$$\lim_{\epsilon \downarrow 0} \frac{2\pi}{\epsilon} P_{\sigma_{1-\epsilon}}(\theta \leq \Theta) = \frac{8}{\theta}.$$

We now get that

$$\begin{aligned} \frac{2\pi}{\epsilon} P_{\sigma_{1-\epsilon}}(\theta \leq \Theta) &= \frac{2\pi}{\epsilon} \int_{\partial B(0, 1-\epsilon)} P_z(\theta \leq \Theta) \sigma_{1-\epsilon}(dz) \\ &= \frac{2\pi}{\epsilon} P_{1-\epsilon}(\theta \leq \Theta) = \frac{2\pi}{\epsilon} \int_{r\sqrt{t} \geq \theta} f_{|\log(1-\epsilon)|}(t)h(r)dt dr, \end{aligned}$$

where we used rotational invariance in the second equality and Lemma 5.1 in the last equality. We have

$$\frac{2\pi}{\epsilon} \int_{r\sqrt{t} \geq \theta} f_{|\log(1-\epsilon)|}(t) h(r) dt dr \stackrel{(64)}{=} \frac{-\log(1-\epsilon)}{\epsilon} \int_{r\sqrt{t} \geq \theta} e^{-\log^2(1-\epsilon)/2t} \sqrt{\frac{2\pi}{t^3}} h(r) dt dr.$$

Note that  $-\log(1-\epsilon)/\epsilon \rightarrow 1$  as  $\epsilon \rightarrow 0$ ,  $e^{-\log^2(1-\epsilon)/2t}$  is monotone in  $\epsilon$ , and  $e^{-\log^2(1-\epsilon)/2t} \rightarrow 1$  as  $\epsilon \rightarrow 0$  for  $t > 0$ . Hence, the monotone convergence theorem gives that

$$\mu(A_\theta) = \lim_{\epsilon \downarrow 0} \frac{2\pi}{\epsilon} \int_{r\sqrt{t} \geq \theta} f_{|\log(1-\epsilon)|}(t) h(r) dt dr \stackrel{(64)}{=} \int_{r\sqrt{t} \geq \theta} \sqrt{\frac{2\pi}{t^3}} h(r) dr dt. \quad (72)$$

This integral is easily computed as

$$\int_{r\sqrt{t} \geq \theta} \sqrt{\frac{2\pi}{t^3}} h(r) dr dt = \sqrt{2\pi} \int_0^\infty \int_{t \geq (\theta/r)^2} \frac{1}{t^{3/2}} dt h(r) dr \quad (73)$$

$$= \sqrt{2\pi} \frac{2}{\theta} \int_0^\infty r h(r) dr = \sqrt{2\pi} \frac{2}{\theta} E[R(1)] \stackrel{(63)}{=} \frac{8}{\theta}, \quad (74)$$

finishing the proof of the lemma.  $\square$

**Remark.** Lemma 5.2 implies that  $\mu(A_{2\pi}) = 4/\pi$ . Hence, under  $\mathbb{P}_{\mathbb{D}}$ ,  $\omega_\alpha(A_{2\pi})$  is a Poisson random variable with mean  $\alpha 4/\pi$ . In particular,

$$\mathbb{P}_{\mathbb{D}}(\omega_\alpha(A_{2\pi}) = 0) > 0. \quad (75)$$

We will now use Lemma 5.2 to prove Theorem 2.3.

*Proof of Theorem 2.3.* Define the measure  $m$  on  $(0, 2\pi)$  by letting

$$m(A) = \int_A \frac{8}{t^2} dt, \quad A \in \mathcal{B}((0, 2\pi)).$$

Lemma 5.2 implies that under  $\mathbb{P}_{\mathbb{D}}$ ,

$$\Xi_\alpha \text{ is a Poisson point process on } (0, 2\pi) \text{ with intensity measure } \alpha m. \quad (76)$$

We now consider the Poisson point process on  $(1/(2\pi), \infty)$  defined by

$$\Xi_\alpha^{-1} := \sum_{(w_i, \alpha_i) \in \text{supp}(\omega_\alpha)} \delta_{\Theta(w_i)^{-1}}. \quad (77)$$

Now note that for  $1/(2\pi) < t_1 < t_2$  we have that

$$m([1/t_2, 1/t_1]) = 8(t_2 - t_1).$$

Hence,  $\Xi_\alpha^{-1}$  is a homogeneous Poisson point process on  $(1/(2\pi), \infty)$  with intensity  $8\alpha$ . Now let  $\Delta_1 = 1/\Theta_{(1), \alpha}$  and for  $n \geq 2$  let

$$\Delta_n := 1/\Theta_{(n), \alpha} - 1/\Theta_{(n-1), \alpha}.$$

Then  $\Delta_n$  is a sequence of i.i.d. exponential random variables, with mean  $1/(8\alpha)$ . Since

$$1/\Theta_{(n),\alpha} = \sum_{i=1}^n \Delta_i,$$

we get that

$$P\left(\left|\frac{1}{\Theta_{(n),\alpha}} - \frac{n}{8\alpha}\right| > n^{3/4} \text{ i.o.}\right) = 0. \quad (78)$$

Since

$$\left|\Theta_{(n),\alpha} - \frac{8\alpha}{n}\right| = \left|\frac{1/\Theta_{(n),\alpha} - n/(8\alpha)}{n/(8\alpha\Theta_{(n),\alpha})}\right|,$$

and  $1/\Theta_{(n),\alpha} > cn$  for all but finitely many  $n$  for some constant  $c > 0$  a.s., Equation (78) implies that for some constant  $c'(\alpha) < \infty$ ,

$$P\left(\left|\Theta_{(n),\alpha} - \frac{8\alpha}{n}\right| \leq c'(\alpha)n^{-5/4} \text{ for all but finitely many } n\right) = 1. \quad (79)$$

Let  $Y_{n,\alpha} = \sum_{i=1}^n \Theta_{(i),\alpha} - \sum_{i=1}^n \frac{8\alpha}{i}$ . From (79) and the triangle inequality we see that a.s.,  $Y_{\infty,\alpha} := \lim_{n \rightarrow \infty} Y_{n,\alpha}$  exists and  $|Y_{\infty,\alpha}| < \infty$  a.s. Hence,

$$\tilde{Y}_{\infty,\alpha} := \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{\Theta_{(i),\alpha}}{2\pi} - \frac{4\alpha \log(n)}{\pi}\right)$$

exists and is finite a.s. Hence, the sum in (62) is finite a.s. if  $\alpha < \pi/4$  and infinite a.s. if  $\alpha \geq \pi/4$ . Let  $\tilde{V}_{\infty}^{\alpha}$  denote the event that there is some  $\theta \in [0, 2\pi)$  such that  $[0, e^{i\theta})$  intersects only a finite number of trajectories in the support of  $\omega_{\alpha}$ . The above, together with (75), shows that  $\mathbb{P}_{\mathbb{D}}(\tilde{V}_{\infty}^{\alpha}) = 1$  if  $\alpha < \pi/4$  and  $\mathbb{P}_{\mathbb{D}}(\tilde{V}_{\infty}^{\alpha}) = 0$  if  $\alpha \geq \pi/4$ . It remains to argue that that  $\mathbb{P}_{\mathbb{D}}(V_{\infty}^{\alpha}) > 0$  when  $\alpha < \pi/4$ . So now fix  $\alpha < \pi/4$ . Let  $\tilde{V}_{\infty,R}^{\alpha}$  be the event that there is some  $\theta \in [0, 2\pi)$  such that  $[0, e^{i\theta})$  intersects only trajectories in the support of  $\omega_{\alpha}$  which also intersect the ball  $B(o, R)$ . If  $\tilde{V}_{\infty}^{\alpha}$  occurs, then for some random  $R_0 < 1$ , the event  $\tilde{V}_{\infty,R}^{\alpha}$  occurs for every  $R \in (R_0, 1)$ . Hence for some  $R_1 < 1$ ,  $\mathbb{P}_{\mathbb{D}}(\tilde{V}_{\infty,R_1}^{\alpha}) > 0$ . Suppose that  $\tilde{\omega} \in \Omega_{\mathbb{D}}$  and write

$$\hat{\omega}_{\alpha} = 1_{W_{B(0,R_1)}} \tilde{\omega}_{\alpha} + 1_{W_{B(0,R_1)}^c} \omega_{\alpha}.$$

Observe that if  $\omega_{\alpha} \in \tilde{V}_{\infty,R_1}^{\alpha}$  and  $\tilde{\omega}_{\alpha}(W_{B(0,R_1)}) = 0$ , then  $\hat{\omega}_{\alpha} \in V_{\infty}^{\alpha}$ . Hence

$$\mathbb{P}_{\mathbb{D}}^{\otimes 2}(\hat{\omega}_{\alpha} \in V_{\infty}^{\alpha}) \geq \mathbb{P}_{\mathbb{D}}(\tilde{V}_{\infty,R_1}^{\alpha})\mathbb{P}_{\mathbb{D}}(\tilde{\omega}_{\alpha}(W_{B(0,R_1)}) = 0) > 0.$$

The result follows, since  $\omega_{\alpha}$  under  $\mathbb{P}_{\mathbb{D}}$  has the same law as  $\hat{\omega}_{\alpha}$  under  $\mathbb{P}_{\mathbb{D}}^{\otimes 2}$ .  $\square$

## References

- [1] Benjamini, I., Jonasson, J., Schramm, O. & Tykesson, J. (2009). Visibility to infinity in the hyperbolic plane, despite obstacles. *Alea*, 6, 323–342.
- [2] BreLOT, M., GowriSankaran, K. & Murthy, M.V. (1967). *Lectures on potential theory*, vol. 19. Tata Institute of Fundamental Research Bombay.
- [3] Calka, P., Michel, J. & Porret-Blanc, S. (2009). Asymptotics of the visibility function in the boolean model. *arXiv preprint arXiv:0905.4874*.
- [4] Cerný, J.r. & Popov, S. (2012). On the internal distance in the interlacement set. *Electron. J. Probab.*, 17, no. 29, 25.
- [5] Feller, W. (1951). The asymptotic distribution of the range of sums of independent random variables. *Ann. Math. Statistics*, 22, 427–432.
- [6] Grigor'yan, A. (1999). Analytic and geometric background of recurrence and non-explosion of the brownian motion on riemannian manifolds. *Bulletin of the American Mathematical Society*, 36, 135–249.
- [7] Hsu, E.P. (2002). *Stochastic analysis on manifolds*, vol. 38. American Mathematical Soc.
- [8] Kahane, J.P. (1990). Recouvrements aléatoires et théorie du potentiel. *Colloq. Math.*, 60/61, 387–411.
- [9] Kahane, J.P. (1991). Produits de poids aléatoires indépendants et applications. In *Fractal geometry and analysis (Montreal, PQ, 1989)*, vol. 346 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 277–324, Kluwer Acad. Publ., Dordrecht.
- [10] LacoIn, H. & Tykesson, J. (2013). On the easiest way to connect  $k$  points in the random interlacements process. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10, 505–524.
- [11] Lawler, G.F. (2005). *Conformally invariant processes in the plane*, vol. 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.
- [12] Lawler, G.F. & Werner, W. (2000). Universality for conformally invariant intersection exponents. *J. Eur. Math. Soc. (JEMS)*, 2, 291–328.
- [13] Lawler, G.F. & Werner, W. (2004). The Brownian loop soup. *Probab. Theory Related Fields*, 128, 565–588.
- [14] Li, X. (2016). Percolative properties of brownian interlacements and its vacant set. *arXiv preprint arXiv:1610.08204*.
- [15] Lyons, R. (1996). Diffusions and random shadows in negatively curved manifolds. *J. Funct. Anal.*, 138, 426–448.

- [16] Mörters, P. & Peres, Y. (2010). *Brownian motion*, vol. 30. Cambridge University Press.
- [17] Pólya, G. (1918). Zahlentheoretisches und wahrscheinlichkeitstheoretisches über die sichtweite im walde. *Arch. Math. Phys*, 27, 135–142.
- [18] Port, S.C.J., Sidney C. (1978). *Brownian motion and classical potential theory*. Academic Press New York.
- [19] Procaccia, E.B. & Tykesson, J. (2011). Geometry of the random interlacement. *Electron. Commun. Probab.*, 16, 528–544.
- [20] Ráth, B. & Sapozhnikov, A. (2012). Connectivity properties of random interlacement and intersection of random walks. *ALEA Lat. Am. J. Probab. Math. Stat.*, 9, 67–83.
- [21] Shepp, L.A. (1972). Covering the circle with random arcs. *Israel J. Math.*, 11, 328–345.
- [22] Sidoravicius, V. & Sznitman, A.S. (2009). Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.*, 62, 831–858.
- [23] Siegel, A.F. & Holst, L. (1982). Covering the circle with random arcs of random sizes. *J. Appl. Probab.*, 19, 373–381.
- [24] Sznitman, A.S. (2010). Vacant set of random interlacements and percolation. *Ann. of Math. (2)*, 171, 2039–2087.
- [25] Sznitman, A.S. (2013). On scaling limits and brownian interlacements. *Bulletin of the Brazilian Mathematical Society, New Series*, 44, 555–592.
- [26] Thäle, C. (2014). Hausdorff dimension of visible sets for well-behaved continuum percolation in the hyperbolic plane. *Braz. J. Probab. Stat.*, 28, 73–82.
- [27] Tykesson, J. (2008). *Continuum percolation in non-Euclidean spaces*. Chalmers University of Technology.
- [28] Tykesson, J. & Windisch, D. (2012). Percolation in the vacant set of Poisson cylinders. *Probab. Theory Related Fields*, 154, 165–191.
- [29] Virág, B. (2003). Brownian beads. *Probab. Theory Related Fields*, 127, 367–387.
- [30] Wu, H. (2012). On the occupation times of Brownian excursions and Brownian loops. In *Séminaire de Probabilités XLIV*, vol. 2046 of *Lecture Notes in Math.*, 149–166, Springer, Heidelberg.