On the relationships between the geometric and the algebraic ideas in Duhre's textbooks of mathematics, as reflected via Book II of Euclid's *Elements*

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Abstract

The present article explores the relationships between the geometric and algebraic ideas presented in Anders Gabriel Duhre's mathematics textbooks. Of particular interest is Book II of Euclid's Elements *as presented by Duhre in his textbook on geometry from 1721. We consider in detail Duhre's two versions of Proposition II.5, dealing with straight lines cut into equal and unequal parts, as well as the two proofs of the propositions that he presents. Duhre's formulations are slightly different from traditional geometric formulations, as he moved away from a purely geometrical context towards an algebraic one. Duhre established Proposition II.5 using algebra in Descartes' notation as well as in the notation of Wallis and Oughtred. Duhre´s reason for introducing algebra in Book II of Euclid's* Elements *was to obtain convenience in calculations, as well as the possibility to generalize results to different kinds of quantities.*

Introduction

Anders Gabriel Duhre (1680–1739, or possibly 1681–1739) was a Swedish mathematician and mathematics teacher. He was the son of the Circuit judge Gabriel Duhre in Waksala outside of Uppsala. In 1695 he became a student at Uppsala University, where he studied for the astronomy professor Pehr Elvius (1660–1718), but he left the university during the year of the plague 1710. In 1712 he was a student of the Swedish scientist, inventor and industrialist Christopher Polhem (1661–1751) at his school Laboratorium Mechanicum in Stjärnsund. For a few years he then taught mathematics to engineering students at Bergskollegium – a central agency with the task to lead and control the mining and metal processing – and to prospective officers at the Royal Fortification Office in Stockholm. In 1723, after receiving permission from the parliament, Duhre opened his own school, Laboratorium Mathematico-Oeconomicum, in Ultuna outside Uppsala. This school is the precursor to the Swedish University of Agricultural Sciences, which is located in the same area. Duhre's school was a technical school, economically based on farming operations, where young boys were taught theoretical and practical subjects. Of particular interest is that mathematics was taught in this school; for example, it is known that infinitesimal calculus was for the first time taught in Sweden in Duhre's school. The

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mathematics teaching at the school was not located at Ultuna, but at Uppsala, close to the university (Rodhe, 2002).

Even though Duhre never received a position at Uppsala University, he was an important and influential person within the Swedish mathematical society. He had knowledge of modern mathematics that was not taught at the university, and among his students were several of the mathematicians to be established during the 1720s and 1730s – among others Eric Burman (1692–1729), Anders Celsius (1701–1744) and Mårten Strömer (1707–1770) – all of whom would later become professors of mathematics. As students at Uppsala University they turned to Duhre to learn more on modern mathematics, which they apparently did not have the opportunity to do at the university. Samuel Klingenstierna (1698–1765), the most important and internationally most well-known Swedish mathematician during the 18th century, was probably not a student of Duhre, but he was recommended by Duhre to study, among others, Charles Reyneau's (1656–1728) book *Analyse demontrée* (1708) on differential and integral calculus (Rodhe, 2002).

Duhre taught in Swedish and planned early on to write mathematical textbooks in Swedish, in order to introduce the Swedish youth to the new and modern mathematics. He contributed with two textbooks in mathematics – one in algebra and one in geometry. The first book, *En Grundelig Inledning til Mathesin Universalem och Algebram* (A Thorough Introduction to Universal Mathematics and Algebra), was edited by his student Georg Brandt (1694–1768) and published in 1718. In this textbook, which is based on Duhre's notes from his lectures at Bergskollegium, modern algebra according to René Descartes' (1596–1650) notation is presented, as well as examples from Isaac Newton's (1642–1727), John Wallis' (1616–1703), and Bernard Nieuwentijt's (1654–1718) theories from the late 17th century.

The second textbook, *Första Delen af en Grundad Geometria* (The first part of a founded geometry), was published in 1721 and was based on Duhre's lectures held in Swedish at the Royal Fortification Office. He probably planned a second book on geometry, but this was never realized. Duhre's book on geometry is the most advanced textbook in mathematics in Swedish during the 18th century. It is a voluminous book of about 600 pages, which distinguishes itself from previous books on geometry by not being based on Euclid's *Elements*. Instead, most of the book treats infinitely large and infinitely small quantities. Duhre also takes advantage of the theories he earlier presented in his book on algebra. Of particular interest in his book on geometry is his use of algebra in the geometrical context as presented via parts of Book II of Euclid's *Elements*.

After war, plague and bad harvest, Sweden in the 1710s was a devastated country in great need of supply, science and new ideas. Duhre was convinced that knowledge of the new mathematics, together with the physics derived therefrom, would provide an increased prosperity to the country. In the introduction to his book on geometry, Duhre wrote that his motive to teach and write in Swedish was to make it possible for talented students, who due to poverty had no experience in foreign languages, to study mathematics (Duhre, 1721). This was an important step to implement Duhre's vision. Probably his two books were used at Bergskollegium and at the Fortification office at least until 1723 when he opened his own school. It is not known if the books were used at Laboratorium Mathematico-Oeconomicum, even though it is likely that some of the modern mathematical ideas presented in his books were also taught at this school.

Duhre's school project ended in personal disaster when he in 1731 had to leave the school. By economical prompting the governor Johan Brauner (1668–1743) managed to get the Parliament to transfer the lease of the farm, where the school was located, to him. Duhre was left in poverty, but he still had ideas on educational initiatives, and wanted to start similar schools in the whole country. This was however not realized and he died in 1739 (Hebbe, 1933).

Duhre was the precursor of many modern ideas in mathematics as well as in the technical education and in the rationalization of farming. His work contributed to the constitution of the professorship of economics at Uppsala University in 1740 (Hebbe, 1933). Duhre is known as a great inspirer, and due to his teaching and his two books on algebra and geometry, Swedish mathematics in the 1730s had become just as advanced as in most countries in Europe (Rodhe, 2002).

Book II of Euclid's *Elements* and the relationships between geometry and algebra

Book II of the *Elements* attributed to Euclid contains 14 propositions on plane geometry and it raises interesting questions regarding the relationships between geometry and algebra. For example, during the 1970s there was a rather heated debate about whether the Greeks presented a kind of algebra in some of their geometry, and especially Book II of Euclid's *Elements* was discussed as an example of Greek algebra hidden behind a "geometrical veil". In 1975 Sabetai Unguru argued that the claim that Euclid was a "geometric algebraist", handling geometrical notions but actually practicing common algebra, was incorrect and based on an anachronistic reading of ancient Greek texts in the sense that they were translated into a modern algebraic notation; according to Unguru algebra was imposed on the Greek texts rather than discovered in them. In 1978, the leading mathematician André Weil dismissed Unguru's critique by accusing Unguru of not knowing enough mathematics, claiming, without much justification, that Euclid just used a somewhat cumbersome notation in his algebra. Nowadays Weil's claim is instead sometimes regarded as a scandal in the field of the history of mathematics (Öberg, 2011, p. xxv; Corry, 2013, p. 638).

In a paper presented at the HPM 2008 satellite meeting of ICME 11 in Mexico City, Gert Schubring discussed the use of historical material in the teaching of mathematics. He exploited the debate on the existence of "geometric algebra" in Greek mathematics provoked by Unguru, to initiate a methodological debate on the use of sources that have been modernized and distorted for didactical reasons. For several essential reasons, such as for example conceptualization, notation, language and epistemology, this modification of sources constitutes a common practice in projects making use of history of mathematics in the teaching of mathematics. The question is which degree of distortion can be claimed to be legitimate for the aim of teaching (Schubring, 2008).

In the first chapter of his book on geometry Duhre stated and proved eight of the propositions of Book II of Euclid's *Elements*; he did however not include the first two and the last four of the propositions attributed to Euclid. Later in the same chapter he stated the propositions again, now also including the first two, in an alternative way. This is probably the first time parts of the *Elements* were published in the Swedish language. However, Duhre has earlier not been acknowledged for the publishing of the first Swedish edition of parts of Euclid's *Elements*. Previously the first Swedish edition has been attributed to Duhre's student Mårten Strömer (see, for example, Heath 1956, p. 113). In 1744 Strömer published a Swedish translation of the first six books of Euclid's *Elements*, in a traditional geometrical context.

Nevertheless, neither Duhre nor Strömer were the first to publish the *Elements* in Sweden. Already in 1637 the Swedish mathematician Martinus Erici Gestrinius (1594–1648) had contributed with a commented edition of the *Elements* in Latin. Gestrinius did include algebra into his geometry, at least in Propositions 4, 5 and 6 of Book II. He did this by associating the propositions with three different kinds of quadratic equations, before showing how the equations can be solved in three different ways: rhetorically, with tables, and geometrically. Thus, Gestrinius used the quadratic equations to illustrate the propositions; that is, he utilized algebra in order to illustrate geometry (Pejlare & Rodhe, 2016). Duhre probably had studied Gestrinius' edition of the *Elements*, since it was used at Uppsala University. Also Christopher Clavius' edition of the *Elements* from 1574 was used at Uppsala University, and was most likely known by Duhre. The wording of the propositions and proofs of Book II of Clavius and Gestrinius are very similar, but Clavius did not include any algebra. However, as we will see in the following section, Duhre's presentation of the propositions and proofs of Book II is very different from Gestrinius' version, as well as from Clavius' version, and the traditional geometrical formulations and proofs of Euclid.

Formulation and proof of Proposition II.5

In order to illustrate how Duhres' formulations of the propositions of Book II, as well as his proofs, differ from Euclid's, we will investigate one of the propositions in detail. The proposition we will consider is Proposition II.5, which in Duhre's edition is the third proposition, dealing with straight lines cut into equal and unequal parts. A traditional formulation, attributed to Euclid, of this proposition is as follows:

Proposition II.5: If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equal the square on the half (Heath 1956, p.382).

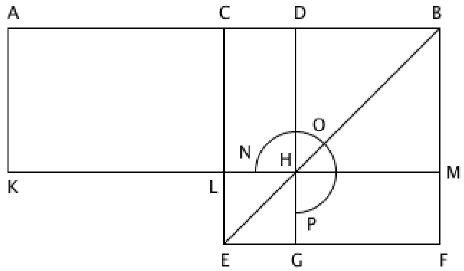


Fig. 1. A visualization of Euclid's Proposition II.5

According to the proposition and referring to Fig. 1, the straight line AB is cut into equal segments at C and into unequal segments at D. The rectangle $ADHK$ together with the square $LHGE$ equal the square $CBFE$. We can easily approve this proposition, as the segment AC equals the segment BF and the segment AK equals the segment BD and thus the rectangle $ACLK$ equals the rectangle $BFGD$.

Duhre's first formulation of Proposition II.5, translated into English, is as follows:

Duhre´s first version of Proposition II.5: If something whole is divided into two equal parts and then into two unequal parts, then the *product* of the unequal parts together with the *square* of the difference between one of the equal and one of unequal parts is equal to the *square* of the half of the whole (Duhre 1721, p. 20).[1]

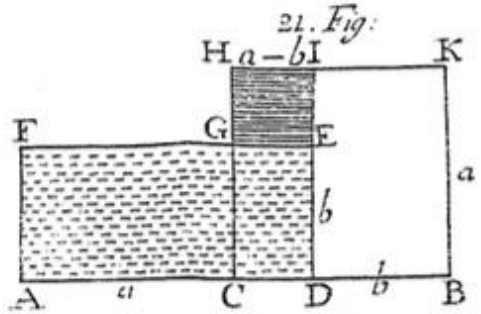
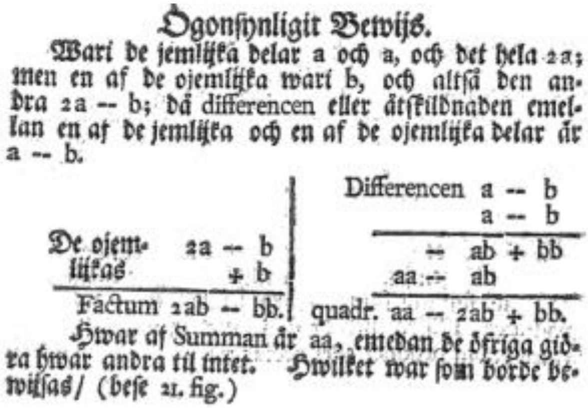


Fig. 2. Duhre's visualization of Proposition II.5 (Duhre 1721, p. II).

[1] "Om något helt warder fördelat uti twänne jemlijka delar och der näst uti twänne andra ojemlijka måtte de ojemlijkas *product*, jemte *quadraten* af den åtskilnad som är emillan en af de jemlijka och en af de ojemlijka delar wara jemlijk emot *quadraten* af bemelte helas halfpart."

As we can see, Duhre's formulation of the proposition is slightly different from the traditional formulation. When Euclid used the concepts *straight line*, *segment*, *rectangle,* and *square on the straight line*, Duhre used the concepts *something whole*, *part*, *product,* and *square of the difference*. This indicates that Duhre moved away from a purely geometrical context and considered the proposition also in an algebraic context. This becomes even clearer as we consider Duhre's proof of the proposition. The traditional proof of Euclid is purely geometrical, but Duhre's version is, even if it refers to Fig. 2 above, purely algebraical (see Fig. 3):


Ögonskynligit Bewijs.

Wari de jemlijka delar a och a, och det hela 2a; men en af de ojemlijka wari b, och altså den andra 2a -- b; då differencen eller åtskildnaden emellan en af de jemlijka och en af de ojemlijka delar är a -- b.

De ojemlijkas	2a -- b	Differencen a -- b
	+ b	a -- b
		-- ab + bb
		aa -- ab
	Factum 2ab -- bb.	quadr. aa -- 2ab + bb.

Hwar af Summan är aa, emedan de öfriga giöra hwar andra til intet. Hwilket war som borde bewisas/ (bese 21. fig.)

Fig. 3. Duhre's proof of Proposition II.5 (Duhre, 1721, p. 20).

Duhre lets the equal parts be a, the whole is $2a$ and one of the unequal parts is b. Using algebra he now shows that $(2a - b)b + (a - b)^2 = a^2$.

This proof is very different from the geometrical one we know from Euclid. Instead, Duhre established the proposition, as well as the remaining seven propositions he included in his book, using algebraic ideas in Descartes' notation. Thus, Duhre's proof could be seen as a proof of an algebraic identity where he performs operations on algebraic expressions. Duhre motivated his choice of using algebra in the following manner:

> Here would have been an opportunity to prove the preceding propositions according to Euclid, which is both certain and beautiful; but as the *Method* to prove through symbols is more *universal* such that it for the sense reveals the unchangeable truth of these propositions, with the assurance that they do not only refer to *lines*, but also to *surfaces*, solids, and everything that belongs to the word *quantities*; thus the great advantage that therein consists can be observed (Duhre, 1721, p. 24).[2]

[2] "Här hade man fuller haft tilfälle at efter Euclidis sätt bewijsa föregående förestälningar hwilket är både säkert och wackert; men såsom *Methoden* at bewijsa igienom kiennetekn är mera *universal*, så at den för förnuftet uppenbarar dessa förestälningars oföränderliga wisshet med försäkran at de icke allenast sträkia sig til *linier*, utan och jämwäl til *superficier*, fasta kroppar och alt det som hörer under ordet *quantum*; altså kan man förmärkia den stora fördehl som der uti består."

Reading this quote it becomes clear that Duhre knew of the Euclidean geometrical proofs of the propositions of Book II, but he thought that this method – to use algebra – is much more general since the quantity he refers to as *something whole* does not have to be a straight line but could also be another kind of quantity, such as a surface, a solid, or something else. Thus, even though he refers to a figure (Fig. 2) where *something whole* actually is considered to be a straight line, this does not have to be the case.

Proposition II.5 in Wallis' and Oughtred's notation

William Oughtred (1574–1660) was one of the first mathematicians to exemplify theorems of classic geometry using algebra (Stedall, 2002). He demonstrated all of the 14 propositions of Book II of Euclid's *Elements* in his *Clavis mathematicæ* from 1631 with his analytical method, which means that he used François Viète's (1540–1603) algebraic notation, as presented in Viète's symbolic algebra, or the *Analytical Art*. During the end of the 16th century Viète was inspired by Diophanto's work and used capital letters instead of abbreviations as symbols for the unknown and known entities.

Duhre formulated and proved eight of the propositions of Book II of the *Elements* using algebra in Descarte's notation before he mentions John Wallis and William Oughtred. Duhre refers to Chapter 26 of Wallis' *A Treatise of Algebra* from 1685, where Wallis used Oughtred's notation to present the first ten of the 14 propositions of Book II of Euclid's *Elements*. Duhre considers this notation to be both "clear and convenient for the sense" (Duhre, 1721, p. 26), and thus he proceeds in presenting these 10 propositions of Book II in a similar way as Wallis had done. Duhre's second formulation of Proposition II.5 – in Wallis' and Oughtred's notation and translated into English – is as follows:

> **Duhre's second version of Proposition II.5:** If a straight *line* such as AB is distributed into two equal parts AC, BC and into two unequal parts AD, BD, (that is $z = 2S = a + e$) then the rest that remains when from the square of the half the *rectangle*, or the *oblong*, contained by the unequal parts has been removed be equal to the *square* of the middle part CD, that is $S^2 - ae = Q{:}S - e = Q{:}a - S = Q{:}\frac{1}{2}x$ (Duhre, 1721, p. 27).[3]

Thus, Duhre lets the straight line $AB = z$ be distributed into two equal parts S and into the two unequal parts a and e, that is, $z = 2S = a + e$. The symbol Q stands for the squaring of the expression, that is $Q{:}S - e = (S - e)^2$. The proposition claims that the rest that remains when the rectangle contained by the unequal parts has been removed from the square of the half, that is $S^2 - ae$, equals the square of the middle part, that is $(S - e)^2 = (a - S)^2 = (\frac{1}{2}x)^2$, where $x = a - e$ is twice the middle part.

[3] "Om en rät *linea* såsom denna AB, warder fördelad uti twänne jemlijka delar AC, BC, och sedan uti twänne andra ojemlijka AD, BD, (det är $z = 2S = a + e$) måtte resten som öfwerblifwer sedan man från hälftens quadrat borttagit *rectangeln*, eller *oblongen*, innesluten af de ojemlijka delar wara jemlijk emot *quadraten* af millanstycket CD, nemligen $s^2 - ae = Q{:}S - e = Q{:}a - S = Q{:}\frac{1}{2}x$."

We notice that Duhre's language has changed in this second presentation of the proposition. When he in his first presentation uses the concepts *something whole* and *product*, he in this latter presentation uses the concepts *straight line* and *rectangle*. This indicates that Duhre now moves back towards a more geometrical understanding of the proposition. Nevertheless, he still uses the concept *part* instead of the concept *segment*, illustrating a difference from the geometrical understanding of the proposition attributed to Euclid. Moreover, in Duhre's second proof of the proposition an algebraic language is used.

We will now consider Duhre's second proof of Proposition II.5 (see Fig. 4), in which he utilized the notations of Wallis and Oughtred:


C D
5. Om en rät linea såsom denna A|--|--|-|B, warder fördelad uti twänne jemlikka delar/ AC, BC, och sedan uti twänne andra ojemlikka/ AD, BD, (det är z = 2S = a + e) måtte resten som öfwerblifwer/ sedan man ifrån hälftens quadrat borttagit rectangeln, eller oblongen, innesluten af de ojemlikka delar/ wara jemlikt emot quadraten af mällanstycket/ CD, nemligen S: ae = Q:S -- e = Q:a -- S = Q: ½ x, det är/ om z = a + e och x = a -- e, at ¼ z² -- ¼ x² måtte wara = ae. Ty om quadraten af a + e warder beteknad med Q: a + e; af a -- e med Q: a -- e, och man ifrån z² (= Q: a + e) = a² + e² + 2ae, borttager x² (= Q: a -- e) = a² + e² -- 2ae. blifwer Resten . . . . z² -- x² = 4 ae.
¼z² -- ¼x² = ae.
S² -- ae = ¼z² -- ae = ¼x²
= Q: ½x = Q: ½z -- e =
Q: a -- ½z.
D 2

Fig. 4. Duhre's formulation and proof of Proposition II.5, utilizing Wallis' and Oughtred's notation (Duhre, 1721, p. 27)

First Duhre shows that $\frac{1}{4}z^2 - \frac{1}{4}x^2 = ae$. He does this by using $z = a + e$ and $x = a - e$ and calculating

$$Q: a + e = a^2 + e^2 + 2ae \quad \text{and} \quad Q: a - e = a^2 + e^2 - 2ae.$$

This implies that

$$z^2 - x^2 = a^2 + e^2 + 2ae - (a^2 + e^2 - 2ae) = 4ae.$$

Thus $\frac{1}{4}z^2 - \frac{1}{4}x^2 = ae$, and since $S^2 = \frac{1}{4}z^2$, Duhre now concludes that

$$S^2 - ae = \frac{1}{4}z^2 - ae = \frac{1}{4}x^2 = Q: \frac{1}{2}x = Q: \frac{1}{2}z - e = Q: a - \frac{1}{2}z,$$

which establishes the proposition.

Even though Duhre in the formulation of the proposition partly relied on a geometrical interpretation, the proof he performed is entirely algebraic. Duhre claims that he follows Wallis and in fact he uses some of the symbols that Wallis used in *A Treatise of algebra* in 1685. For example Wallis, as well as Duhre, used the symbol Q to indicate the squaring of an expression. Duhre also used the same letters as Wallis, even though Wallis used capital letters and Duhre usually used lowercase letters.

Concluding remarks

Duhre was primarily an educator and his importance in the Swedish history of mathematics lies in his ability to transferring modern mathematics to the following generation of Swedish mathematicians. His textbooks on algebra and geometry were written in Swedish, which was important to reach a wider audience in Sweden.

One interesting question is why Duhre used algebra in his presentation of Book II of Euclid's *Elements*. With algebra Duhre could obtain convenience in calculations, since complicated expressions can be transformed into simpler ones. With algebra geometrical results can also be generalized to different kinds of quantities, since unknowns do not necessarily have to be, for example, lines. Throughout his book on geometry Duhre gave many examples of how algebra can be used to solve geometrical problems. The book is concluded with the following statement:

> Now it is unnecessary to give more examples to demonstrate the usefulness of algebra in geometry, and how those in previous chapters given linear demonstrations can easily be shown by algebra; Therefore there is every reason to do so. (Duhre, 1721, p. 561)[4]

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[4] "Nu synes onödigt at med flera exempel wijsa Algebrans nytta wid Rätliniska Geometrien, samt huru de i föregående Capitlen stående lineariska demonstrationer lätteligen igenom Algebra kunna uplösas; Emedan man af dessa der til kan hafwa en fullkomlig Anledning."

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