Further results on consensus formation in the Deffuant model[*]

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Abstract

The so-called Deffuant model describes a pattern for social interaction, in which two neighboring individuals randomly meet and share their opinions on a certain topic, if their discrepancy is not beyond a given threshold θ. The major focus of the analyses, both theoretical and based on simulations, lies on whether these single interactions lead to a global consensus in the long run or not. First, we generalize a result of Lanchier for the Deffuant model on $\mathbb{Z}$, determining the critical value for θ at which a phase transition of the long term behavior takes place, to other distributions of the initial opinions than i.i.d. uniform on $[0, 1]$. Then we shed light on the situations where the underlying line graph $\mathbb{Z}$ is replaced by higher-dimensional lattices $\mathbb{Z}^d$, $d \geq 2$, or the infinite cluster of supercritical i.i.d. bond percolation on these lattices.

1 Introduction

Let $G = (V, E)$ be a simple graph, i.e. having undirected edges and neither loops nor multiple edges. The considered graph may either be finite or infinite with bounded maximal degree. Furthermore, without loss of generality we can assume G to be connected, since in what follows one could consider the connected components seperately otherwise. Every vertex is understood to represent an individual and will at each time $t \geq 0$ be assigned a value representing its opinion. All the edges in E are connections between individuals allowing for mutual influence. There are a number of models for what is called *opinion dynamics*, which are qualitatively different but share similar ideas, see [2] for an extensive survey.

The *Deffuant model* (introduced by Deffuant et al. [3]) featuring two model parameters $\mu \in (0, \frac{1}{2}]$ and $\theta \in (0, \infty)$ is defined as follows. At time $t = 0$, the vertices are assigned i.i.d. initial opinions, in the standard case uniformly distributed on the interval $[0, 1]$. In

addition, serving as a regime for the random encounters, every edge $e \in E$ is assigned a unit rate Poisson process. The latter are independent of each other and the initial distribution of opinion values. Denote the opinion value at $v \in V$ at time t by $\eta_t(v)$, which remains unchanged until at some time t a Poisson event occurs at an edges incident to v, say $e = \langle u, v \rangle$. The opinion values of u and v just before this happens may be $\eta_{t-}(u) = \lim_{s \uparrow t} \eta_s(u) =: a$ and $\eta_{t-}(v) = \lim_{s \uparrow t} \eta_s(v) =: b$ respectively.

If these values are within the confidence bound θ, they come symmetrically closer to each other, if not they stay unchanged, i.e.

$$\eta_t(u) = \begin{cases} a + \mu(b-a) & \text{if } |a-b| \leq \theta, \\ a & \text{otherwise} \end{cases}$$

and similarly (1.1)

$$\eta_t(v) = \begin{cases} b + \mu(a-b) & \text{if } |a-b| \leq \theta, \\ b & \text{otherwise.} \end{cases}$$

Observe that μ is modelling the willingness of the individuals to step towards other opinions encountered that fall within their interval of tolerance, shaped by θ. In other words, a value of μ close to 0 represents a strong reluctance to change one's mind. For the process to be well-defined, on the one hand one has to make sure that neither two Poisson events occur simultaneously nor that there is a limit point in time for the events occuring on edges incident to one fixed vertex. But since the maximal degree is bounded and we assume the vertex set to be countable, this is almost surely the case. On the other hand, there is a more subtle issue in how the simple interactions shape transitions of the whole system on an infinite graph – is it well-defined there as well? For infinite graphs with bounded degree, this problem is settled by standard techniques in the theory of interacting particle systems, see Thm. 3.9 on p. 27 in [11].

The most natural question to ask seems to be, if the individual opinions will converge to a common consensus in the long run or if they are going to be split up into groups of individuals holding different opinions. In this regard let us define the following types of scenarios for the asymptotic behavior of the Deffuant model on a connected graph as $t \to \infty$:

Definition 1.1.

(i) *No consensus*
   There will be finally blocked edges, i.e. edges $e = \langle u, v \rangle$ s.t.

$$|\eta_t(u) - \eta_t(v)| > \theta,$$

   for all times t large enough. Hence the vertices fall into different opinion groups.

(ii) *Weak consensus*
   Every pair of neighbors $\{u, v\}$ will finally concur, i.e.

$$\lim_{t \to \infty} |\eta_t(u) - \eta_t(v)| = 0.$$

(iii) *Strong consensus*
   The value at every vertex converges, as $t \to \infty$, to a common limit l, where

$$l = \begin{cases} \text{the average of the initial opinion values,} & \text{if } G \text{ is finite} \\ \mathbb{E}\, \eta_0, & \text{if } G \text{ is infinite.} \end{cases}$$

Let the scenario in which we have weak consensus, but at some vertices v the value $\eta_t(v)$ is not converging be called strictly weak consensus. Whether strictly weak consensus can actually occur (for some graphs and some initial distributions) is an open problem.

On finite graphs, strictly weak consensus is impossible as the opinion average is preserved over time and in general the answer to the question whether we get consensus in the long run or not clearly depends on the initial setting. With independent initial opinions distributed uniformly on $[0,1]$ even for values of θ close to but smaller than 1 consensus might be prevented, albeit with a small probability, e.g. when we get stuck right from the beginning with all the opinions being close to either 0 or 1 leaving a gap larger than θ in between, preventing any two individuals situated at different ends of the opinion range from compromising. In the interdisciplinary area labelled "socio-physics" some work has been done in simulating the long-term behavior of this model on various types of finite graphs, such as in [15].

On infinite regular lattices however, the picture is different and the minimal example almost settled. For the graph on $\mathbb{Z}$ in which consecutive integers are joined by edges, Lanchier [10] showed for the standard case with i.i.d. unif($[0,1]$) distributed initial values that regardless of μ, which is just controlling the speed of convergence, the threshold between no consensus and consensus θ_c is $\frac{1}{2}$, which is the essence of Theorem 2.1.

In this paper, we investigate what happens when this basic setting is generalized, in two different directions. In Section 2 we stay on the one-dimensional lattice, i.e. the line graph on $\mathbb{Z}$, but allow for more general initial distributions and are able to settle most but not all cases of i.i.d. initial configurations (see Theorem 2.2). We also generalize the model slightly to allow for dependent initial opinions given by stationary ergodic sequences that satisfy the so-called *finite energy condition*, known from percolation theory. (The generalization of the Deffuant model to multivariate opinions can be found in the upcoming paper [7].)

In Section 3, $\mathbb{Z}$ is replaced by the general regular lattice $\mathbb{Z}^d$. For $d \geq 2$ most of the techniques developed for the one-dimensional case $\mathbb{Z}$ break down, but we are at least able to show that there won't be disagreement for a sufficiently large confidence bound, larger than $\frac{3}{4}$ in the standard i.i.d. uniform case (see Theorem 3.1). Furthermore, the arguments used transfer with only minor changes to the more general case of an infinite, locally finite, transitive and amenable graph (see Remark 3.6).

Finally, in the last section we consider the Deffuant model on the random subgraph of $\mathbb{Z}^d$ given by supercritical i.i.d. bond percolation independent of the random variables driving the opinion dynamics, i.e. the initial configuration and the Poisson processes. Besides an extension of the result we derived for the full grid to this setting (Theorem 4.2), a lower bound for values of θ allowing for strong consensus on the infinite component is established (Theorem 4.3).

We find it slightly surprising that we can prove this last result for supercritical percolation (with $p < 1$) but not for the full lattice. The more common situation for random processes living on supercritical percolation clusters is that these are easier to handle on the full lattice.

2 Generalized initial configurations on $\mathbb{Z}$

2.1 Independent and identically distributed initial opinion values

Theorem 2.1 (Lanchier). *Consider the Deffuant model on the graph $(\mathbb{Z}, E)$, where $E = \{\langle v, v+1 \rangle, v \in \mathbb{Z}\}$ with i.i.d. unif($[0,1]$) initial configuration and fixed $\mu \in (0, \frac{1}{2}]$.*

(i) *If $\theta > \frac{1}{2}$, the model converges almost surely to strong consensus, i.e. with probability 1 we have: $\lim_{t \to \infty} \eta_t(v) = \frac{1}{2}$ for all $v \in \mathbb{Z}$.*

(ii) *If $\theta < \frac{1}{2}$ however, the integers a.s. split into (infinitely many) finite clusters of neighboring individuals asymptotically agreeing with one another, but no global consensus is approached.*

For the line graph, the critical value θ_c equals thus $\frac{1}{2}$, but what happens at criticality is still an open question. Lanchier's result was reproven by Häggström using somewhat more basic techniques (see [5], Thm. 6.5 and Thm. 5.2).

It turns out that the methods in [5] can be adapted to i.i.d. initial distributions beyond the unif($[0,1]$) case. In the following theorem, we determine θ_c in all cases except when the distribution's positive and negative parts both have infinite expectation (this case remains unsolved). Upon completing this work, we learned that a similar extension was simultaneously and independently done by Shang [14]. Part (a) of our Theorem 2.2 conflicts with Thm. 1 in [14], the discrepancy being due to Shang overlooking the crucial effect that gaps in the support of the distribution of η_0 have, if they are large.

Theorem 2.2. *Consider the Deffuant model on $\mathbb{Z}$ as described earlier with the only exception that the initial opinions are not necessarily distributed uniformly on $[0,1]$ (but still i.i.d.).*

(a) *Suppose the initial opinion of all the agents follows an arbitrary bounded distribution $\mathcal{L}(\eta_0)$ with expected value $\mathbb{E}\,\eta_0$ and $[a,b]$ being the smallest closed interval containing its support. If $\mathbb{E}\,\eta_0$ does not lie in the support, there exists some maximal, open interval $I \subset [a,b]$ such that $\mathbb{E}\,\eta_0$ lies in I and $\mathbb{P}(\eta_0 \in I) = 0$. In this case let h denote the length of I, otherwise set $h = 0$.*

   Then the critical value for θ, where a phase transition from a.s. no consensus to a.s. strong consensus takes place, becomes $\theta_c = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}$. The limit value in the supercritical regime is $\mathbb{E}\,\eta_0$.

(b) *Suppose the initial opinions' distribution is unbounded but its expected value exists, either in the strong sense, i.e. $\mathbb{E}\,\eta_0 \in \mathbb{R}$, or the weak sense, i.e. $\mathbb{E}\,\eta_0 \in \{-\infty, +\infty\}$. Then the Deffuant model with arbitrary fixed parameter $\theta \in (0, \infty)$ will a.s. behave subcritically, meaning that no consensus will be approached in the long run.*

Before embarking on the proof of this generalized result, let us recall some key ingredients of the proof for the standard uniform case in [5]. The arguably most central among these is the idea of *flat points*. A vertex $v \in \mathbb{Z}$ is called ε *-flat to the right* in the initial configuration $\{\eta_0(u)\}_{u \in \mathbb{Z}}$ if for all $n \geq 0$:

$$\frac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u) \in \left[\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right]. \tag{2.1}$$

It is called ε-*flat to the left* if the above condition is met with the sum running from $v - n$ to v instead. Finally, v is called *two-sidedly ε-flat* if for all $m, n \geq 0$

$$\frac{1}{m+n+1} \sum_{u=v-m}^{v+n} \eta_0(u) \in \left[\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right]. \tag{2.2}$$

In order to grasp the crucial role of flat points another concept has to be mentioned, namely the representation of $\eta_t(v)$ as a weighted average of initial opinions (see La. 3.1 in [5]). This convex combination of initial opinions can be written in a neat form, using as a tool the non-random pairwise averaging procedure Häggström called *Sharing a*

drink (SAD) in [5]. In the latter, one has an initial profile $\{\xi_0(v)\}_{v \in \mathbb{Z}}$, with $\xi_0(0) = 1$ and $\xi_0(v) = 0$ for all $v \neq 0$, symbolizing a full glass of water at site 0 and empty ones at all other sites. The averaging is now done as in (1.1) but without the threshold θ and the encounters are no longer random, but given by a sequence of edges. Elements of $[0, 1]^{\mathbb{Z}}$ that can be obtained by a finite such sequence are called SAD-profiles. An appropriately tailored SAD-procedure will then mimick the dynamics of the corresponding Deffuant model backwards in time in such a way that the state $\eta_t(0)$ in the Deffuant model at any given time $t > 0$ can be written as a weighted average of states at time 0 with weights given by an SAD-profile. In [5], general properties of SAD-profiles and consequences for $\eta_t(0)$ are derived. For example, the opinion value at a vertex which is two-sidedly ε-flat in the initial configuration can throughout time not move further away than 7ε from its initial value (see La. 6.3 in [5]).

Proof of Theorem 2.2. (a) The proof of this part will be subdivided into three steps marked by (i), (ii) and (iii).

(i) At first, let us suppose that the initial opinions are distributed on $[0, 1]$ according to $\mathcal{L}(\eta_0)$ having expected value $\mathbb{E}\,\eta_0 = \frac{1}{2}$ and mass around the expectation as well as at least one of the extremes, i.e. for all $\varepsilon > 0$ we have

$$\mathbb{P}\left(\eta_0 < \varepsilon \text{ or } \eta_0 > 1 - \varepsilon\right) > 0, \quad \mathbb{P}\left(\tfrac{1}{2} - \varepsilon \leq \eta_0 \leq \tfrac{1}{2} + \varepsilon\right) > 0.$$

Then we claim that the result of Theorem 2.1 still holds true.

To prove this generalization of the standard uniform case is in fact to check that the crucial conditions in Häggström's [5] proof are met. First of all, the i.i.d. property guarantees that the distribution of the initial configuration is translation invariant, hence both the left- and right-shift of the system (that is $v \mapsto v - 1 \,\forall v \in \mathbb{Z}$ and $v \mapsto v + 1 \,\forall v \in \mathbb{Z}$ respectively) are measure-preserving.

The proof of La. 4.2 in [5] showing that $\mathbb{P}(v \text{ is } \varepsilon\text{-flat to the right}) > 0$ for every $\varepsilon > 0$ and $v \in \mathbb{Z}$ only uses the Strong Law of Large Numbers (SLLN), local modification (which employs that $\mathbb{P}\left(\tfrac{1}{2} - \varepsilon \leq \eta_0(v) \leq \tfrac{1}{2} + \varepsilon\right) > 0$ for all $\varepsilon > 0$, which we assumed) as well as $\mathbb{E}\,\eta_0 = \frac{1}{2}$.

By symmetry the same is true for ε-flatness to the left and the additional assumption that $\mathbb{P}(\eta_0 \notin [\varepsilon, 1 - \varepsilon]) > 0$ provides the missing ingredient to mimick Prop. 5.1 and Thm. 5.2 in [5] verbatim: If $\theta < \frac{1}{2}$, pick $\varepsilon > 0$ small enough such that $\theta \leq \frac{1}{2} - 2\varepsilon$. With positive probability any given site v is prevented from ever compromising with its neighbors already by the initial configuration, namely if $v - 1$ is ε-flat to the left, $v + 1$ ε-flat to the right and v itself an outlier in the sense that $\eta_0(v) \notin [\varepsilon, 1 - \varepsilon]$. This establishes the subcritical case *(i)* in Theorem 2.1.

To show $\mathbb{P}(v \text{ is two-sidedly } \varepsilon\text{-flat}) > 0$ for all $v \in \mathbb{Z}$, $\varepsilon > 0$ (in La. 4.3 in [5]) it is used once more that $\mathbb{P}\left(\tfrac{1}{2} - \varepsilon \leq \eta_0 \leq \tfrac{1}{2} + \varepsilon\right) > 0$. Following the reasoning of Sect. 6 in [5] literally will settle the supercritical case. The only change that has to be made in order to adapt to the generalized setting is that the expected energy at time $t = 0$, i.e. $\mathbb{E}\left(\eta_0(v)^2\right) \in (0, 1]$ in La. 6.2, is no longer $\frac{1}{3}$ as for the uniform distribution. This minor change is not crucial however, since only the value's finiteness is used in the proof of Prop. 6.1.

(ii) Now suppose the initial distribution is as in (i), but fails to have mass around the expectation $\frac{1}{2}$ and leaves a gap of width $h \in (0, 1]$, i.e. there exists some maximal (open) interval $I \subset [0, 1]$ of length h such that $\frac{1}{2}$ lies inside I and $\mathbb{P}(\eta_0 \in I) = 0$. Then we claim that the critical value becomes $\theta_{\mathrm{c}} = \max\{\frac{1}{2}, h\}$.

Changing the assumptions concerning the initial distribution of opinions as in (ii) will affect both the sub- and supercritical case as outlined in step (i). Clearly, the limiting behavior a.s. cannot be consensus for $\theta < h$ due to the fact that with probability 1 we will have initial opinion values both below and above $\frac{1}{2}$. Since an update, according to (1.1), can only take place between neighbors that are either both below or both above $\frac{1}{2}$, sites with initial values above the gap I will throughout time stay above it and the same holds for initial values below the gap. In particular, edges that are blocked due to incident values lying on different sides of the gap I in the beginning will stay blocked for ever, making consensus impossible.

For $\theta > h$, however, the behavior is pretty much as in the first case. Nevertheless, when it comes to show that there will be arbitrarily flat points with positive probability, one has to go about somewhat differently due to the fact that for sufficiently small ε, $\mathbb{P}\left(\eta_0 \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]\right) = 0$, which implies that no site can be ε-flat in the initial configuration by the very definition of flatness (taking $n = 0$ in (2.1) and $m = n = 0$ in (2.2) respectively).

Let the gap interval be denoted by $I = (\alpha, \alpha + h)$ and fix $\delta > 0$. Choose two rational numbers in $[0, \frac{1}{2}) \cap [\alpha - \delta, \alpha]$ and $(\frac{1}{2}, 1] \cap [\alpha + h, \alpha + h + \delta]$ respectively, say p and q, and define $I_1 := [p, \alpha]$ and $I_2 := [\alpha + h, q]$. Since I is maximal, one can choose these rationals in such a way that

$$\mathbb{P}(\eta_0 \in I_1) > 0 \text{ as well as } \mathbb{P}(\eta_0 \in I_2) > 0.$$

Clearly, there exist natural numbers m, n s.t. $\frac{m}{m+n} p + \frac{n}{m+n} q = \frac{1}{2}$. As numbers from I_1 and I_2 differ not more than δ from p and q respectively, the average of m numbers from I_1 and n numbers from I_2 surely lies within $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$.
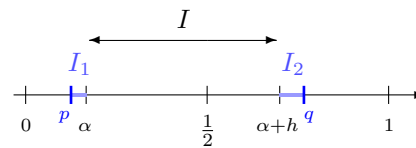
Thus, we get that for any fixed $k \in \mathbb{N} = \{1, 2, \dots\}$:

$$\mathbb{P}\left(\frac{1}{k(m+n)} \sum_{v=0}^{k(m+n)-1} \eta_0(v) \in \left[\tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta\right]\right) > 0. \qquad (2.3)$$

Now let us consider some fixed time point $t > 0$ and the corresponding configuration $\{\eta_t(v)\}_{v \in \mathbb{Z}}$. There is a.s. an infinite increasing sequence of not necessarily consecutive edges $(\langle v_k, v_k + 1 \rangle)_{k \in \mathbb{N}}$ to the right of site 0, on which no Poisson event has occurred up to time t.

Clearly, their positions are random, so let $l_k := v_{k+1} - v_k$, for $k \in \mathbb{N}$, denote the random lengths of the intervals in between and $l_0 := v_1 - v_0 + 1$ the one of the interval including 0, where $\langle v_0 - 1, v_0 \rangle$ is the first edge to the left of the origin without Poisson event. Since the involved Poisson processes are independent, it is easy to verify that the l_k, $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, are i.i.d., having a geometric distribution on $\mathbb{N}$ with parameter e^{-t}.

For $\delta > 0$, let A_δ be the event that l_0 is finite and only finitely many of the events $\{l_k \geq k\delta\}$, $k \in \mathbb{N}$, occur. Then their independence and the Borel-Cantelli-Lemma tell us that A_δ has probability 1. On A_δ however the following holds a.s. true:

$$\limsup_{v \to \infty} \frac{1}{v+1} \sum_{u=0}^{v} \eta_t(u) = \limsup_{v \to \infty} \frac{1}{v+1} \sum_{u=v_0}^{v} \eta_t(u)$$

$$\leq \limsup_{v \to \infty} \frac{1}{v+1} \sum_{u=v_0}^{v} \eta_0(u) + \delta$$

$$= \lim_{v \to \infty} \frac{1}{v+1} \sum_{u=0}^{v} \eta_0(u) + \delta = \frac{1}{2} + \delta.$$

The inequality follows from the fact that the Deffuant model is mass-preserving in the sense that $\eta_t(u) + \eta_t(v) = \eta_{t-}(u) + \eta_{t-}(v)$ in (1.1), hence for all $k \in \mathbb{N}$: $\sum_{u=v_0}^{v_k} \eta_0(u) = \sum_{u=v_0}^{v_k} \eta_t(u)$. For the average at time t running from v_0 to some $v \in \{v_k + 1, \ldots, v_{k+1}\}$ to differ by more than δ from the one at time 0, the interval has to be of length more than $k\delta$, since $v_k \geq k$ and $\eta_t(u) \in [0, 1]$ for all t, u. This, however, will happen only finitely many times. Since δ was arbitrary and mimicking the same argument for the limes inferior, we have established that

$$\lim_{v \to \infty} \frac{1}{v+1} \sum_{u=0}^{v} \eta_t(u) = \frac{1}{2} \text{ almost surely.} \qquad (2.4)$$

Now fix $\varepsilon > 0$ such that $h + \frac{\varepsilon}{3} < \theta$, choose $\delta = \frac{\varepsilon}{6}$ in (2.3) as well as the rationals p, q and integers m, n accordingly. Due to (2.4) there exists some integer number k s.t. the event

$$A := \left\{ \frac{1}{v+1} \sum_{u=0}^{v} \eta_t(u) \in \left[\frac{1}{2} - \frac{\varepsilon}{3}, \frac{1}{2} + \frac{\varepsilon}{3} \right] \text{ for all } v \geq N \right\}$$

has probability greater than $1 - \mathrm{e}^{-2t}$, where $N := k(m+n) - 1$. Let B in turn be the event that there was no Poisson event on $\langle -1, 0 \rangle$ and $\langle N, N+1 \rangle$ up to time t, hence $\mathbb{P}(B) = \mathrm{e}^{-2t}$. Finally, let C be the event that the initial values $\eta_0(0), \ldots, \eta_0(N)$ were all in $[p, q]$, km of them below $\frac{1}{2}$, kn above $\frac{1}{2}$, and the Poisson firings on the edges $\langle 0, 1 \rangle, \ldots, \langle N-1, N \rangle$ up to time t are sufficiently numerous such that, given B, $\eta_t(u) \in [\frac{1}{2} - \frac{\varepsilon}{3}, \frac{1}{2} + \frac{\varepsilon}{3}]$ for all $u \in \{0, \ldots, N\}$. Note that $q - p \leq h + 2\delta < \theta$, hence every such Poisson event will lead to an update, and that the independence of the initial configuration and the Poisson processes together with the considerations leading to (2.3) imply that C has positive probability. Furthermore, C is independent of B and $A \cap B$ cannot have probability 0, since

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) > (1 - \mathrm{e}^{-2t}) + \mathrm{e}^{-2t} - \mathbb{P}(A \cup B) \geq 0.$$

This gives that the conditional probabilities $\mathbb{P}(A|B)$ and $\mathbb{P}(C|B)$ are both strictly greater than 0.

Given B, we can apply the coupling trick, commonly known as *local modification*, precisely as in the proof of La. 4.2 in [5] to find that $\mathbb{P}(A \cap B \cap C) > 0$. A one-line calculation shows that $A \cap B \cap C$ implies the ε-flatness to the right of site 0 in the configuration at time t.

Since the distribution of $\{\eta_t(u)\}_{u \in \mathbb{Z}}$ is still translation and left-right reflection invariant, every site $v \in \mathbb{Z}$ is ε-flat to the right (or left) at time t with positive probability on the one hand, and on the other this allows us to follow the argument in (i) settling the subcritical case and forcing $\theta_c \geq \max\{\frac{1}{2}, h\}$.

A short moment's thought verifies that ε-flatness to the right of site v and ε-flatness to the left of site $v - 1$ simultaneously imply two-sided ε-flatness of both, v and $v - 1$.

Let A_v^r, B_v^r, C_v^r be the sets appearing above, corresponding to site v and "right", and $A_{v-1}^l, B_{v-1}^l, C_{v-1}^l$ the ones corresponding to $v-1$ and "left". The involved independences lead to

$$\mathbb{P}(A_v^r \cap B_v^r \cap C_v^r \cap A_{v-1}^l \cap B_{v-1}^l \cap C_{v-1}^l)$$
$$= \mathbb{P}(A_v^r \cap C_v^r \cap A_{v-1}^l \cap C_{v-1}^l | B_v^r \cap B_{v-1}^l) \cdot \mathbb{P}(B_v^r \cap B_{v-1}^l)$$
$$= \mathbb{P}(A_v^r \cap C_v^r | B_v^r \cap B_{v-1}^l) \cdot \mathbb{P}(A_{v-1}^l \cap C_{v-1}^l | B_v^r \cap B_{v-1}^l) \cdot \mathbb{P}(B_v^r \cap B_{v-1}^l)$$
$$= \mathbb{P}(A_v^r \cap C_v^r | B_v^r) \cdot \mathbb{P}(A_{v-1}^l \cap C_{v-1}^l | B_{v-1}^l) \cdot \mathbb{P}(B_v^r \cap B_{v-1}^l) > 0,$$

since $\mathbb{P}(B_v^r \cap B_{v-1}^l) = e^{-3t} > 0$. Hence two-sided ε-flatness at time t has positive probability as well. Following the argument corresponding to the supercritical case in (i), using the preserved translation invariance of the distribution of $\{\eta_t(u)\}_{u \in \mathbb{Z}}$ once more, we find that there will be consensus in the long run, if only $\theta > \max\{\frac{1}{2}, h\}$.

Putting both arguments together, this proves the claim $\theta_c = \max\{\frac{1}{2}, h\}$.

(iii) Finally, suppose that $[a, b]$ is the smallest closed interval containing the support of the initial opinions' distribution and that the latter features a gap of width $h \in [0, b-a]$ around the expected value $\mathbb{E}\,\eta_0 \in [a, b]$. Then we claim that the critical value becomes $\theta_c = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}$ and the limit in the case of strong consensus is $\mathbb{E}\,\eta_0$.

Clearly, the dynamics of the Deffuant model are not effected by translations of the initial distribution ($x \mapsto x + c$ for some constant $c \in \mathbb{R}$). A scaling ($x \mapsto \frac{x}{c}$, $c \in \mathbb{R}_{>0}$) has the only effect that the value for the parameter θ has to be rescaled too, in order to get identical dynamics.

Let $c := \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0\}$ and consider the linear transformation

$$x \mapsto \frac{x - \mathbb{E}\,\eta_0}{2\,c} + \frac{1}{2}.$$

The transformed initial distribution satisfies the assumptions in step (ii) and leaves a gap of width $\frac{h}{2c}$ around the mean $\frac{1}{2}$. Therefore, the considerations in (ii) allow us to conclude

$$\theta_c = 2\,c \cdot \max\{\tfrac{1}{2}, \tfrac{h}{2c}\} = \max\{c, h\} = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}.$$

Note that the limit of an individual opinion in the supercritical case is the retransformed equivalent of $\frac{1}{2}$, i.e. $2\,c \cdot \left(\frac{1}{2} + \left(\frac{\mathbb{E}\,\eta_0}{2\,c} - \frac{1}{2}\right)\right) = \mathbb{E}\,\eta_0$.

(b) To prove the statement on unbounded initial distributions we have to treat two cases, namely the one where $\mathbb{E}\,|\eta_0| < \infty$ and the other where exactly one of both $\mathbb{E}\,\eta_0^+, \mathbb{E}\,\eta_0^-$ is infinite.

(i) In case of an unbounded initial distribution with existing first moment and expectation $\mathbb{E}\,\eta_0 < \infty$, the SLLN reads (for arbitrarily chosen $v \in \mathbb{Z}$):

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u) = \mathbb{E}\,\eta_0\right) = 1.$$

Consequently, there exists some number $r > 0$ s.t.

$$\mathbb{P}\left(\frac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u) \in [\mathbb{E}\,\eta_0 - r, \mathbb{E}\,\eta_0 + r] \text{ for all } n \in \mathbb{N}_0\right) > 0.$$

Slightly abusing the definition (the expectation $\frac{1}{2}$ in (2.1) would have to be replaced by $\mathbb{E}\,\eta_0$), one could say that with positive probability site v is r-flat to the right.

Let the confidence bound θ take on some value in $(0, \infty)$. Strictly along the lines of Prop. 5.1 in [5], it follows that if $v-1$ and $v+1$ are r-flat to the left and right respectively and simultaneously $\eta_0(v) \notin [\mathbb{E}\,\eta_0 - r - \theta, \mathbb{E}\,\eta_0 + r + \theta]$ – an event with positive probability – the values at $v-1$ and $v+1$ will throughout all of time stay within the interval $[\mathbb{E}\,\eta_0 - r, \mathbb{E}\,\eta_0 + r]$ leaving the edges $\langle v-1, v \rangle$ and $\langle v, v+1 \rangle$ blocked. Since this happens at every site v with positive probability, ergodic theory tells us that it will almost surely occur at infinitely many sites.

(ii) Now suppose that the expectation of η_0 exists only in the weak sense, i.e. $\mathbb{E}\,\eta_0 \in \{-\infty, +\infty\}$. Once more, symmetry allows us to focus on the case $\mathbb{E}\,\eta_0^+ = \infty$, $\mathbb{E}\,\eta_0^- < \infty$. In this case the SLLN reads

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{1}{n} \sum_{u=v+1}^{v+n} \eta_0(u) = \infty\right) = 1. \qquad (2.5)$$

We can assume $\mathbb{P}(\eta_0 < 0) > 0$, otherwise a translation (irrelevant for the dynamics) as in the last step of (a) will reduce the problem to this setting. Some one-sided version of the idea of proof using flatness can then be employed.

Let the confidence bound $\theta \in (0, \infty)$ be arbitrary but fixed. By (2.5), for sufficiently large $N \in \mathbb{N}$ the following event has non-zero probability:

$$A_N := \left\{\frac{1}{n} \sum_{u=v+1}^{v+n} \eta_0(u) > \theta \text{ for all } n \geq N\right\}.$$

Local modification is again the key step to advance. Let $\xi := \mathcal{L}(\eta_0)$ denote the distribution of η_0 and $\xi|_{(\theta, \infty)}$ its distribution conditioned on the event $\{\eta_0 > \theta\}$. Clearly, ξ is stochastically dominated by $\xi|_{(\theta, \infty)}$, i.e. $\xi \preceq \xi|_{(\theta, \infty)}$, implying

$$\mathcal{L}\big((\eta_0(u))_{u \geq v+1}\big) = \bigotimes_{u \geq v+1} \xi \preceq \left(\bigotimes_{u=v+1}^{v+N} \xi|_{(\theta, \infty)}\right) \otimes \left(\bigotimes_{u > v+N} \xi\right).$$

Let B be the event $\{\eta_0(v+1) > \theta, \ldots, \eta_0(v+N) > \theta\}$, which has non-zero probability, and

$$A_1 := \left\{\frac{1}{n} \sum_{u=v+1}^{v+n} \eta_0(u) > \theta \text{ for all } n \in \mathbb{N}\right\}.$$

The stochastic domination from above yields:

$$\mathbb{P}(A_1) \geq \mathbb{P}(A_1 \cap B) = \mathbb{P}(A_N \cap B) = \mathbb{P}(A_N|B) \cdot \mathbb{P}(B)$$
$$\geq \mathbb{P}(A_N) \cdot \mathbb{P}(B) > 0.$$

The very same ideas as in the proof of Prop. 5.1 in [5] show that if A_1 occurs and the edge $\langle v, v+1 \rangle$ doesn't allow for an update, irrespectively of the dynamics on $\{u \in \mathbb{Z}, u \geq v+1\}$, we have that $\eta_t(v+1) > \theta$ is preserved for all times $t > 0$. By symmetry the same holds for site $v-1$ and the half-line to the left, i.e. $\{u \in \mathbb{Z}, u \leq v-1\}$. Independence of the initial opinions therefore guarantees that with positive probability, the initial configuration can be such that $\eta_0(v) < 0$ and the values at sites $v-1$ and $v+1$ are doomed to stay above θ, blocking the edges adjacent to v once and for all. Ergodicity makes sure that with probability 1 infinitely many sites will get stuck this way. $\qquad \square$

Example 2.3. *(a) As a first toy application of the above result, let us consider the Deffuant model on $\mathbb{Z}$ in which the initial values are independently distributed according to a beta distribution $\text{Beta}(\alpha, \beta)$, where the two real numbers $\alpha, \beta > 0$ represent the parameters of this family of distributions. That means η_0 has support $[0, 1]$ and its distribution the density function*

$$f_{\alpha,\beta}(x) = \frac{1}{\text{B}(\alpha,\beta)} \, x^{\alpha-1} \, (1-x)^{\beta-1}, \quad \text{for } x \in [0, 1],$$

where the normalizing factor is given by the beta function

$$\text{B}(\alpha, \beta) = \int_0^1 t^{\alpha-1} \, (1-t)^{\beta-1} \, dt.$$

Since $f_{\alpha,\beta} > 0$ on the open interval $(0, 1)$, there are no gaps in the support and a simple calculation shows $\mathbb{E}\,\eta_0 = \frac{\alpha}{\alpha+\beta}$. Consequently, part (a) of Theorem 2.2 shows that the critical value for the confidence bound separating the regimes of consensus and fragmentation is

$$\theta_\text{c} = \begin{cases} \frac{\alpha}{\alpha+\beta}, & \text{if } \alpha \geq \beta \\ \frac{\beta}{\alpha+\beta}, & \text{otherwise} \end{cases} = \frac{\max\{\alpha,\beta\}}{\alpha+\beta}.$$

This example appears in [14] as well.

(b) Letting the initial values be independently drawn from a uniform distribution on the discrete set $\{-0.8, -0.3, 0.7, 0.8\}$, $[-0.8, 0.8]$ is the minimal closed interval containing the support of $\mathcal{L}(\eta_0)$. Obviously, there is a gap of width $h = 1$ around the mean $\mathbb{E}\,\eta_0 = 0.1$. Applying part (a) of Theorem 2.2 we can conclude that

$$\theta_\text{c} = \max\{\mathbb{E}\,\eta_0 - (-0.8), 0.8 - \mathbb{E}\,\eta_0, h\} = \max\{0.9, 0.7, 1\} = 1.$$

(c) If we take the initial opinions to be i.i.d. and uniform on the set $[0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$ instead, its expectation is $\mathbb{E}\,\eta_0 = \frac{1}{2}$. But even though $\mathbb{P}(|\eta_0 - \mathbb{E}\,\eta_0| > \frac{1}{2}) = 0$, a choice of $\theta \in (\frac{1}{2}, \frac{3}{4})$ will a.s. lead to no consensus, as $\theta_\text{c} = \frac{3}{4}$, again by part (a) of the above theorem. The next proposition actually shows that even for $\theta = \theta_\text{c}$ the limiting scenario will a.s. be no consensus.

For a bounded initial distribution whose support has a large gap around its mean, we can deal with the behavior at criticality:

Proposition 2.4. *Let the initial opinions be again i.i.d. with $[a, b]$ being the smallest closed interval containing the support of the marginal distribution, and the latter feature a gap (α, β) of width $\beta - \alpha > \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0\}$ around its expected value $\mathbb{E}\,\eta_0 \in [a, b]$.*

At criticality, that is for $\theta = \theta_\text{c} = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, \beta - \alpha\} = \beta - \alpha$, we get the following: If both α and β are atoms of the distribution $\mathcal{L}(\eta_0)$, i.e. $\mathbb{P}(\eta_0 = \alpha) > 0$ and $\mathbb{P}(\eta_0 = \beta) > 0$, the system approaches a.s. strong consensus. However, it will a.s. lead to no consensus if either $\mathbb{P}(\eta_0 = \alpha) = 0$ or $\mathbb{P}(\eta_0 = \beta) = 0$.

Proof. In order to prove this statement, we can follow the arguments in the proof of part (a) of Theorem 2.2. By the translation and scaling invariance of the dynamics as described in step (iii) of the cited proof, we can restrict ourselves to the case in step (ii) and assume that the support of $\mathcal{L}(\eta_0)$ is a subset of $[0, 1]$, $\mathbb{E}\,\eta_0 = \frac{1}{2}$ and $\mathbb{P}(\eta_0 < \varepsilon \text{ or } \eta_0 > 1 - \varepsilon) > 0$ for all $\varepsilon > 0$. Note that under these further assumptions, we have $\theta = \theta_\text{c} = \beta - \alpha > \frac{1}{2}$.

If both ends of the gap are atoms, we can follow the reasoning in the supercritical case in step (ii) of the proof of Theorem 2.2 (a) and for every $\delta > 0$ choose natural numbers m, n such that $\frac{m}{m+n}\alpha + \frac{n}{m+n}\beta \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, to get (2.3). Using such a collection of initial opinions, i.e. m times the value α and n times β, all of them will be precisely within the confidence bound, hence allow for the manipulation described above as local modification. Having arbitrarily flat points with positive probability at time $t > 0$, $\theta > \frac{1}{2}$ guarantees a.s. strong consensus.

The negative statement is easy to handle. If without loss of generality $\mathbb{P}(\eta_0 = \alpha) = 0$, with probability 1 there will be no initial value lying in the interval $[\alpha, \beta]$. Since $\theta = \beta - \alpha$, this gap cannot be bridged. We refer once more to step (ii) in the proof of part (a) of Theorem 2.2 for a more detailed reasoning. $\qquad\square$

Does Proposition 2.4 constitute progress in the attempt to solve the critical case in the setting of uniformly distributed initial opinions (the open problem mentioned right after Theorem 2.1)? Probably not, since in this setting, due to the large width of the gap $\beta - \alpha > \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0\}$, the criticality comes only from the gap in the distribution, not the distance between the mean and the extreme ends of the initial distribution.

As already mentioned in the introductory section, a next step of generalization in terms of the initial opinions would be vector-valued distributions. Despite the fact that this seems to be a minor modification it invokes major changes and would thus excessively expand this section, which is why it is omitted here and treated as a separate topic in [7].

2.2 Dependent initial opinion values

The definition of the Deffuant model generalizes straightforwardly to dependent initial configurations. Considering that – in our treatment of the model on $\mathbb{Z}$ in the foregoing subsection – the independence of initial opinions was merely used to deduce translation invariance and ergodicity with respect to shifts as well as for the local modification, it is a valid question in how far the results of Theorem 2.2 can be generalized to initial configurations $\{\eta_0(v)\}_{v \in \mathbb{Z}}$ that do not form an i.i.d. sequence. The example below shows that stationarity and ergodicity of the sequence of initial opinions is not enough to retain the results from Subsection 2.1. In order to be able to locally modify the configuration as done in the proof of Theorem 2.2, we have to add an extra condition, which is a natural extension to continuous state spaces of the well-known finite energy condition of percolation theory (see for instance Def. 2 in [1]).

Definition 2.5. *Let $\{\xi_v\}_{v \in \mathbb{Z}}$ be a stationary sequence of random variables. It is said to satisfy the* finite energy condition *if it allows conditional probabilities such that the conditional distribution of ξ_0 given $\{\xi_v\}_{v \in \mathbb{Z} \setminus \{0\}}$ almost surely has the same support as the marginal distribution $\mathcal{L}(\xi_0)$.*

Carefully checking its proof with this extra condition in hand, we can get the following generalization of Theorem 2.2:

Theorem 2.6. *Consider the Deffuant model on $\mathbb{Z}$ with initial opinion values $\{\eta_0(v)\}_{v \in \mathbb{Z}}$. If $\{\eta_0(v)\}_{v \in \mathbb{Z}}$ is a stationary sequence of random variables, ergodic with respect to shifts and satisfying the finite energy condition, the results of Theorem 2.2 still hold true.*

To see that the added assumption that conditioning on the configuration apart from a given site v will not change the support of the distribution at site v is essential and can not be dropped, see the following example.

Example 2.7. *Let U be a random variable, uniformly distributed on $\{-4, -3, \ldots, 4\}$. The initial configuration will now be made up of blocks of length 9 centered in the sites*

$\{c_k\}_{k\in\mathbb{Z}} := \{U + 9\,k\}_{k\in\mathbb{Z}}$. *Each block will independently be either of the form $\eta_0(c_k) = \frac{1}{2}$ and $\eta_0(v) = 0$ for $v \in \{c_k - 4, \ldots, c_k - 1, c_k + 1, \ldots, c_k + 4\}$ or $\eta_0(c_k) = \frac{1}{2}$ and $\eta_0(v) = 1$ for $v \in \{c_k - 4, \ldots, c_k - 1, c_k + 1, \ldots, c_k + 4\}$, both with probability $\frac{1}{2}$.*

The initial configuration $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ defined in this way is translation invariant and ergodic with respect to shifts, having the marginal distribution $\mathcal{L}(\eta_0)$ concentrated on $\{0, \frac{1}{2}, 1\}$ with $\mathbb{P}(\eta_0 = 0) = \mathbb{P}(\eta_0 = 1) = \frac{4}{9}$ and $\mathbb{P}(\eta_0 = \frac{1}{2}) = \frac{1}{9}$.

If Theorem 2.1 applied, the critical value should be $\theta_{\mathrm{c}} = \frac{1}{2}$ but it is not hard to see that for $\theta < \frac{4}{5}$ compromises are at first confined to happen within intervals consisting of blocks of the same kind and can thus only lead to values in $[0, \frac{1}{10}] \cup [\frac{9}{10}, 1]$ at sites next to a neighboring block of the other kind, see also Thm. 2.3 in [5]. This means that the edges connecting two blocks of different kind will be blocked throughout time forcing a.s. no consensus.

Due to the fixed block size, the sequence $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ as defined above is obviously not mixing. An easy modification, for instance allowing random block lengths taking values 9 and 11, shows that even an initial configuration which is given by a stationary mixing sequence of random variables does not, in general, allow for the results of the i.i.d. case to be transferred.

3 Upper bound for the critical range of θ on $\mathbb{Z}^d$

3.1 Application of energy arguments

Moving on to higher dimensions as far as the underlying lattice is concerned provides the opportunity to go around blocked edges and there is no handy generalization of the notion of flatness. Among other things, these changes render most of the arguments used in the $\mathbb{Z}$ case void. Enough can be resurrected, however, to establish a lower bound for θ above which consensus is achieved. Throughout Sections 3 and 4 (Theorem 4.3 being an exception) we will only assume that the configuration of initial opinion values $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ is stationary and ergodic with respect to shifts of the kind $T_i : v \mapsto v + e_i$, where e_i is the ith standard basis vector of $\mathbb{R}^d$ for $i \in \{1, \ldots, d\}$.

Theorem 3.1. *Consider the Deffuant model on the d-dimensional lattice $\mathbb{Z}^d$.*

(a) *If the initial values are distributed uniformly on $[0, 1]$ and $\theta > \frac{3}{4}$, the configuration will a.s. approach weak consensus, i.e.*

$$\mathbb{P}\big(\lim_{t\to\infty} |\eta_t(u) - \eta_t(v)| = 0\big) = 1$$

for all $u, v \in \mathbb{Z}^d$ s.t. $\langle u, v \rangle$ forms an edge.

(b) *For general initial distributions on $[0, 1]$ the range of θ, where final consensus is guaranteed, is non-trivial, i.e. including values smaller than 1, unless the initial values are concentrated on 0 and 1, taking on both values with positive probability.*

To prove this, we need first to establish some lemmas, the first one involving the idea of *energy*, introduced in Sect. 6 of [5] (not to be confused with the completely unrelated concept of finite energy from Subsection 2.2).

Assume the initial values $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ have a stationary distribution, ergodic with respect to shifts and the marginal distribution has bounded support, without loss of generality we can take $[0, b]$ to be the smallest closed interval containing it. Denote by $W_t(v) = \mathcal{E}(\eta_t(v))$ the energy at vertex v at time t, where $\mathcal{E} : [0, b] \to \mathbb{R}_{\geq 0}$ is some fixed convex function. If a Poisson event occurs at the edge $e = \langle u, v \rangle$ at time t, and the values at u and v, $\eta_{t-}(u)$ and $\eta_{t-}(v)$ respectively, are within θ, energy is transferred and

(possibly) lost along the edge. The latter to the amount

$$w_t(e) := (W_{t-}(u) + W_{t-}(v)) - (W_t(u) + W_t(v)). \tag{3.1}$$

Since $\eta_t(u) = (1-\mu)\,\eta_{t-}(u) + \mu\,\eta_{t-}(v)$ and $\eta_t(v) = (1-\mu)\,\eta_{t-}(v) + \mu\,\eta_{t-}(u)$, the convexity of $\mathcal{E}$ gives:

$$\begin{aligned}
W_t(u) + W_t(v) &\le (1-\mu)\,W_{t-}(u) + \mu\,W_{t-}(v) + (1-\mu)\,W_{t-}(v) + \mu\,W_{t-}(u) \\
&= W_{t-}(v) + W_{t-}(u),
\end{aligned}$$

i.e. the non-negativity of $w_t(e)$. Let T denote the sequence of arrival times of the Poisson events at e and define the accumulated energy loss along e as

$$W_t^{\mathrm{loss}}(e) := \sum_{s \in T \cap [0,t]} w_s(e).$$

Finally, let $E(v)$ denote the set of edges incident to v and define the total energy attributed to vertex v as

$$W_t^{\mathrm{tot}}(v) := W_t(v) + \frac{1}{2} \sum_{e \in E(v)} W_t^{\mathrm{loss}}(e). \tag{3.2}$$

Note that by (3.1) the sum $W_t^{\mathrm{tot}}(v) + W_t^{\mathrm{tot}}(u)$ is preserved when an update along the edge $\langle u, v \rangle$ takes place. Along the lines of La. 6.2 in [5] we can show the following analog:

Lemma 3.2. *For every $v \in \mathbb{Z}^d$ and $t \ge 0$ we have*

$$\mathbb{E}\left[W_t^{\mathrm{tot}}(v)\right] = \mathbb{E}\left[W_0(\mathbf{0})\right]. \tag{3.3}$$

Proof. Note first that for fixed time t the process $\{W_t^{\mathrm{tot}}(v)\}_{v \in \mathbb{Z}^d}$ only depends on the initial configuration and the independent Poisson processes attributed to the edges. Its distribution is therefore translation invariant and the process ergodic with respect to shifts.

Let $\Lambda_n = [-n, n]^d$ denote the box of sidelength $2n$ centered at the origin $\mathbf{0}$. It contains $|\Lambda_n| = (2n+1)^d$ vertices of the grid $\mathbb{Z}^d$ and there are $2d\,(2n+1)^{d-1}$ edges linking vertices inside Λ_n to vertices outside of the box. The set of such edges is called *edge boundary* of Λ_n and denoted by $\partial_E \Lambda_n$.

The multivariate version of Birkhoff's Theorem, attributed to Zygmund (see e.g. Thm. 10.12 in [8]), tells us that

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{v \in \Lambda_n} W_t^{\mathrm{tot}}(v) = \mathbb{E}\left[W_t^{\mathrm{tot}}(\mathbf{0})\right] \text{ almost surely.} \tag{3.4}$$

Note that the statement of (3.4) is still true if we pass from the original sequence of sets $(\Lambda_n)_{n \in \mathbb{N}}$ to any subsequence.

Translation invariance of the configuration implies $\mathbb{E}\left[W_t^{\mathrm{tot}}(v)\right] = \mathbb{E}\left[W_t^{\mathrm{tot}}(\mathbf{0})\right]$ for all sites v and by definition $W_0^{\mathrm{loss}}(e) = 0$ for all edges e since at time 0 no Poisson event has occurred yet, hence $W_0^{\mathrm{tot}}(\mathbf{0}) = W_0(\mathbf{0})$.

Let us now choose a subsequence $(\Lambda_{n_k})_{k \in \mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \frac{|\partial_E \Lambda_{n_k}|}{|\Lambda_{n_k}|} < \infty. \tag{3.5}$$

As mentioned, (3.4) clearly implies

$$\lim_{k \to \infty} \frac{1}{|\Lambda_{n_k}|} \sum_{v \in \Lambda_{n_k}} W_t^{\mathrm{tot}}(v) = \mathbb{E}\left[W_t^{\mathrm{tot}}(\mathbf{0})\right] \text{ almost surely.} \tag{3.6}$$

In order to establish the claim it is therefore left to show that the limit in (3.6) is constant over time.

Take $\varepsilon > 0$ small and fix a time interval $[t, t + \varepsilon]$. Note that the energy function $\mathcal{E}$ is bounded on $[0, b]$ by $M := \max\{\mathcal{E}(0), \mathcal{E}(b)\}$, due to its convexity. Let $N_{n,\varepsilon}$ be the number of Poisson events on edges in $\partial_E \Lambda_n$ within the time interval $(t, t + \varepsilon]$, see Figure 1, and A_n be the event

$$A_n := \left\{ N_{n,\varepsilon} \geq \tfrac{1}{M} \left(|\partial_E \Lambda_n| + \sqrt{|\Lambda_n|} \right) \right\}.$$

The number on every single edge is a Poisson distributed random variable with parameter ε, consequently having mean and variance ε.

As those random variables are independent, a choice of ε such that $\varepsilon \leq \tfrac{1}{M}$ yields using Chebyshev's inequality:

$$\mathbb{P}(A_n) \leq \mathbb{P}\left(N_{n,\varepsilon} - \mathbb{E}\, N_{n,\varepsilon} \geq \tfrac{1}{M}\sqrt{|\Lambda_n|} \right) \leq M^2\, \frac{\mathrm{var}(N_{n,\varepsilon})}{|\Lambda_n|} \leq M\, \frac{|\partial_E \Lambda_n|}{|\Lambda_n|}.$$
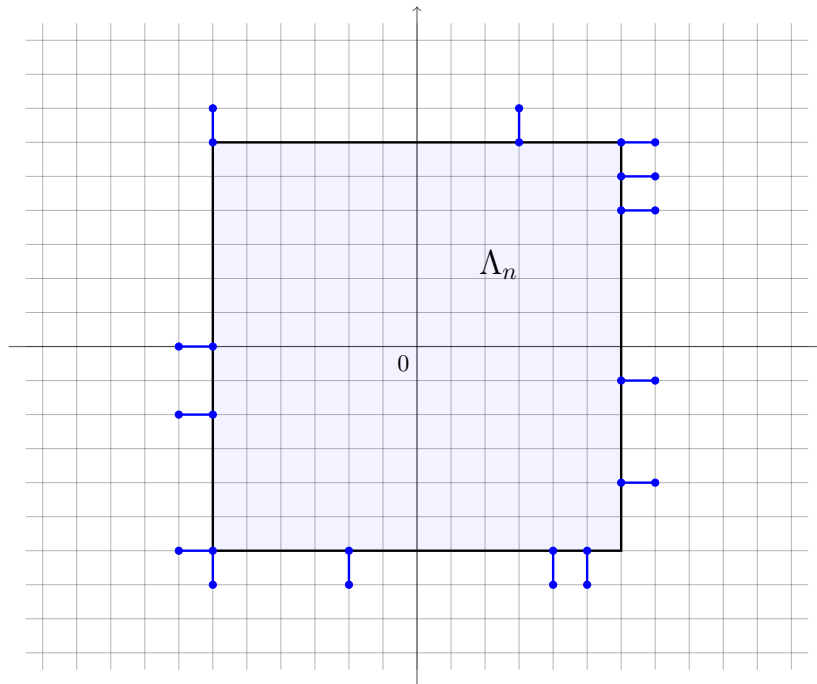

Figure 1: The interactions on the boundary of the box Λ_n in the time interval $[t, t + \varepsilon]$ are few compared to the size of the box for large n.

In view of (3.5), the Borel-Cantelli-Lemma shows that almost surely only finitely many A_{n_k} will occur. In order to conclude, we have to show that this implies

$$\lim_{k \to \infty} \frac{1}{|\Lambda_{n_k}|} \sum_{v \in \Lambda_{n_k}} W_{t+\varepsilon}^{\mathrm{tot}}(v) = \lim_{k \to \infty} \frac{1}{|\Lambda_{n_k}|} \sum_{v \in \Lambda_{n_k}} W_t^{\mathrm{tot}}(v), \qquad (3.7)$$

which in turn guarantees that the limit in (3.6) is constant over time.

It is not hard to convince yourself that Poisson events off $\partial_E \Lambda_{n_k}$ will not change $\sum_{v \in \Lambda_{n_k}} W_t^{\mathrm{tot}}(v)$ and every single event on $\partial_E \Lambda_{n_k}$ can change the sum of total energies

in Λ_{n_k} by at most M. Therefore, on the complement of A_{n_k}, we get that

$$\frac{1}{|\Lambda_{n_k}|} \left| \sum_{v \in \Lambda_{n_k}} W_{t+\varepsilon}^{\mathrm{tot}}(v) - \sum_{v \in \Lambda_{n_k}} W_t^{\mathrm{tot}}(v) \right| \leq \frac{M}{|\Lambda_{n_k}|} \cdot N_{n_k, \varepsilon} < \frac{|\partial_E \Lambda_{n_k}|}{|\Lambda_{n_k}|} + \frac{1}{\sqrt{|\Lambda_{n_k}|}}.$$

As this converges to 0 when $k \to \infty$, we have shown that (3.7) holds almost surely, which concludes the proof. $\qquad\square$

Lemma 3.3. *For the Deffuant model on the lattice $\mathbb{Z}^d$ as above, with threshold parameter $\theta \in (0, b]$, the following holds a.s. for every two neighbors $u, v \in \mathbb{Z}^d$:*

> *Either $|\eta_t(u) - \eta_t(v)| > \theta$ for all sufficiently large t, i.e. the edge $\langle u, v \rangle$*
> *is finally blocked, or* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (3.8)
> $\lim_{t \to \infty} |\eta_t(u) - \eta_t(v)| = 0$, *i.e. the two neighbors will finally concur.*

Proof. The above lemma corresponds to Prop. 6.1 in [5] and the original proof generalizes to the higher-dimensional setting with only minor changes.

As the times between Poisson events on a single edge are exponentially distributed, the memoryless property ensures that given a finite collection of edges and some fixed time s, the edge which experiences the next Poisson event is chosen uniformly at random. Let us take $\mathcal{E} : x \mapsto x^2$ as energy function and fix $e = \langle u, v \rangle$ as well as some $\delta > 0$. If there is a Poisson event at e at time t and the opinion values of u and v are not more than θ apart from each other, energy to the amount of $w_t(e) = 2\mu (1 - \mu)(\eta_{t-}(u) - \eta_{t-}(v))^2$ is lost along the edge, see (3.1). If $|\eta_{t-}(u) - \eta_{t-}(v)| \in (\delta, \theta]$, such an increase of $W_t^{\mathrm{loss}}(e)$ would be at least $2\mu (1 - \mu) \delta^2$. The opinion values of u and v can only change if one of the $4d - 1$ edges incident to either u or v experiences a Poisson event. Given $|\eta_s(u) - \eta_s(v)| \in (\delta, \theta]$ for some fixed time s, the probability that it is in fact e where the first Poisson event after time s on an edge incident to either u or v occurs is $\frac{1}{4d-1}$.

By the extended version of the Borel-Cantelli-Lemma (involving conditional probabilities, see e.g. Cor. 6.20 in [8]) such an increase will happen infinitely often, if $|\eta_t(u) - \eta_t(v)| \in (\delta, \theta]$ for arbitrarily large t, forcing $(W_t^{\mathrm{loss}}(e))_{t \geq 0}$ to diverge. This cannot happen with positive probability, since according to Lemma 3.2 we have

$$\mathbb{E}\left[W_t^{\mathrm{loss}}(e)\right] \leq 2 \, \mathbb{E}\left[W_t^{\mathrm{tot}}(v)\right] = 2 \, \mathbb{E}\left[W_0(\mathbf{0})\right] \leq 2 \, b^2.$$

Hence, it follows that a.s. $|\eta_t(u) - \eta_t(v)| \notin (\delta, \theta]$ for sufficiently large t.

For small values of δ, more precisely $\delta < \frac{\theta}{2}$, the margin $|\eta_t(u) - \eta_t(v)|$ cannot jump back and forth between $[0, \delta]$ and $(\theta, b]$, since single updates can change the value at any site by no more than $\mu\theta \leq \frac{\theta}{2}$. Consequently, for $0 < \delta < \frac{\theta}{2}$, the following holds almost surely:

$$\limsup_{t \to \infty} |\eta_t(u) - \eta_t(v)| \in [0, \delta] \quad \text{or} \quad \liminf_{t \to \infty} |\eta_t(u) - \eta_t(v)| \in (\theta, b].$$

For δ can be chosen arbitrary small and there are only countably many edges, the claim is established. $\qquad\square$

Lemma 3.4. *The probability that there will be finally blocked edges is either 0 or 1.*

Proof. Fix an edge $e = \langle u, v \rangle$ and assume that $\mathbb{P}(e \text{ is finally blocked}) = 0$. By translation invariance of the process, this has to be true for all edges $e \in E$. The union bound together with the preceeding lemma gives:

$$\mathbb{P}\big(\lim_{t \to \infty} |\eta_t(u) - \eta_t(v)| = 0 \; \forall \, u, v \in \mathbb{Z}^d\big) = 1.$$

For $\mathbb{P}(e \text{ is finally blocked}) > 0$, let $N(v)$ denotes the number of edges incident to site v that are finally blocked. Then the ergodicity of $\{\eta_0(v)\}_{v \in \mathbb{Z}^d}$ and the independent Poisson processes attributed to the edges with respect to shifts, forces that almost surely the following holds (using Zygmund's Ergodic Theorem):

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{v \in \Lambda_n} N(v) = \mathbb{E}\left[N(\mathbf{0})\right] = 2d \cdot \mathbb{P}(e \text{ is finally blocked}) > 0.$$

Hence, with probability 1 infinitely many edges will be finally blocked. $\qquad\square$

Having derived these auxiliary results, we can proceed to prove the main result of this section:

Proof of Theorem 3.1. (a) Given some confidence bound $\theta \geq \frac{1}{2}$, the value at every vertex which is incident to a finally blocked edge must be finally located in one of the intervals $[0, 1 - \theta)$ or $(\theta, 1]$. Due to Lemma 3.3 this holds for every vertex almost surely if there are edges which are finally blocked. The foregoing lemma tells us, that if an edge is finally blocked with positive probability, we get

$$\liminf_{t \to \infty} |\eta_t(v) - \tfrac{1}{2}| \geq \theta - \tfrac{1}{2} \quad \text{for all } v \in \mathbb{Z}^d \text{ a.s.} \tag{3.9}$$

Choosing the energy function $\mathcal{E} : x \mapsto |x - \frac{1}{2}|$ and applying Lemma 3.2 we find:

$$\begin{aligned}
\mathbb{E}\left[\liminf_{t \to \infty} W_t(v)\right] &= \mathbb{E}\left[\liminf_{t \to \infty} |\eta_t(v) - \tfrac{1}{2}|\right] \\
&\leq \liminf_{t \to \infty} \mathbb{E}\left[|\eta_t(v) - \tfrac{1}{2}|\right] \\
&\leq \liminf_{t \to \infty} \mathbb{E}\left[W_t^{\text{tot}}(v)\right] \\
&= \mathbb{E}\left[W_0^{\text{tot}}(v)\right] = \tfrac{1}{4},
\end{aligned}$$

where Fatou's Lemma was used in the first inequality and the non-negativity of $W_t^{\text{loss}}(e)$ in the second. If we assume $\mathbb{P}(e \text{ is finally blocked}) > 0$ for some, hence any e, the first expectation must be at least $\theta - \frac{1}{2}$ by (3.9), which leads to a contradiction if θ is larger than $\frac{3}{4}$.

(b) Note that no special feature of $\text{unif}([0,1])$ was used, but $\mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] = \frac{1}{4}$. Consequently, the above result still holds if $\text{unif}([0,1])$ is replaced by some other distribution $\mathcal{L}(\eta_0)$ on $[0,1]$ and the bound $\frac{3}{4}$ replaced by $\mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] + \frac{1}{2}$ simultaneously. Furthermore, this bound is non-trivial, i.e. less than 1, provided $\mathbb{P}(\eta_0 \in \{0,1\}) < 1$ for this implies $\mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] < \frac{1}{2}$. If however $\eta_0 \in \{0,1\}$ almost surely, trivially only $\theta = 1$ will not allow for finally blocked edges, given η_0 is not a.s. constant.

$\qquad\square$

Remark 3.5. *(a) There are two major differences to the results on $\mathbb{Z}$. Firstly, even if intuitively appealing it is no longer ensured that weak consensus as described in Theorem 3.1 will lead to consensus in the strong sense, i.e. that every individual value converges to the mean. By ergodicity we know*

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{v \in \Lambda_n} \mathbb{1}_{\{\lim_{t \to \infty} \eta_t(v) \text{ exists}\}} = \mathbb{P}\big(\lim_{t \to \infty} \eta_t(\mathbf{0}) \text{ exists}\big).$$

In the case of consensus, the indicator functions on the left hand side are either all 0 or all 1. In other words, for θ such that weak consensus is guaranteed, the existence of the limits is an event with probability either 0 or 1. In the latter case

another application of ergodicity and dominated convergence show that this limit must be the mean of the initial distribution:

$$\lim_{t\to\infty} \eta_t(v) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{u\in\Lambda_n} \lim_{t\to\infty} \eta_t(u)$$
$$= \mathbb{E}\left[\lim_{t\to\infty}\eta_t(v)\right]$$
$$= \lim_{t\to\infty}\mathbb{E}\left[\eta_t(v)\right] = \mathbb{E}\left[\eta_0(v)\right],$$

where the first equality follows from weak consensus, the last is Lemma 3.2 with the identity as energy function.

Secondly, it is no longer clear that we can talk about a critical value for θ separating the parameter space neatly into a sub- and a supercritical regime, since final consensus is not necessarily an increasing event in θ. By Lemma 3.4 it is clear that for fixed θ we have that all neighbors finally concur with probability either 0 or 1. Hence both cases can not occur simultaneously but there might be a range for θ in which they alternate, unlike in the case of $\mathbb{Z}$.

(b) *Let us next consider another example. Taking for instance $\mathrm{unif}(\{0, \frac{1}{2}, 1\})$ as distribution of the initial values, the reasoning in part (b) of the theorem shows that finally blocked edges are in this case only possible for*

$$\theta \leq \mathbb{E}\left[|\eta_0 - \tfrac{1}{2}|\right] + \tfrac{1}{2} = \tfrac{1}{3} + \tfrac{1}{2} = \tfrac{5}{6}.$$

For other distributions it might even be beneficial to choose some different convex energy function giving a potentially sharper bound on $\theta \geq \frac{1}{2}$ of the kind: The probability for finally blocked edges can only be non-zero for θ such that

$$\inf\{\mathcal{E}(x),\ x \in [0, 1-\theta) \cup (\theta, 1]\} \leq \mathbb{E}\left[\mathcal{E}(\eta_0)\right].$$

Clearly, this inequality is trivial if the minimal value $\min\{\mathcal{E}(x),\ x \in [0,1]\}$ is attained on $[0, 1-\theta) \cup (\theta, 1]$. If this is not the case, it reads

$$\min\{\mathcal{E}(1-\theta), \mathcal{E}(\theta)\} \leq \mathbb{E}\left[\mathcal{E}(\eta_0)\right], \tag{3.10}$$

due to the convexity of $\mathcal{E}$. Choosing $\mathcal{E}$ such that it vanishes on the support of $\mathcal{L}(\eta_0)$ will only give the trivial bound $\theta \leq \frac{1}{2} + \sup\{|x - \frac{1}{2}|,\ x \in \mathrm{supp}(\mathcal{L}(\eta_0))\}$.

In addition, Jensen's inequality tells us that regardless of the chosen convex energy function, from (3.10) we cannot get a bound on θ so sharp that $\mathbb{E}\,\eta_0 \notin (1-\theta, \theta)$. Since in this case we trivially have

$$\inf\{\mathcal{E}(x),\ x \in [0, 1-\theta) \cup (\theta, 1]\} \leq \mathcal{E}\left(\mathbb{E}\,\eta_0\right) \leq \mathbb{E}\left[\mathcal{E}(\eta_0)\right].$$

Finally, a gap in the distribution of η_0 also reduces the scope of (3.10), since for $\mathbb{P}(\eta_0 \in (1-\theta, \theta)) = 0$ we get:

$$\mathcal{E}(\eta_0) \geq \inf\{\mathcal{E}(x),\ x \in [0, 1-\theta) \cup (\theta, 1]\}\ \text{a.s.}$$

This trivially implies the above inequality.

In summary, the same factors obstructing consensus in the Deffuant model on $\mathbb{Z}$ reappear in this treatment of the higher-dimensional case (cf. part (a) of Theorem 2.2).

(c) Next, it is worth noting that the energy function chosen in the proof of Theorem 3.1 is in fact best possible regarding (3.10) for symmetric distributions. If $\mathcal{E}$ is rescaled by some positive factor or translated by adding a constant, the inequality (3.10) stays unchanged. In the case of a symmetric distribution the inequality is symmetric around $\frac{1}{2}$, which is why it holds for the pair $(x \mapsto \mathcal{E}(x), \theta)$ if and only if it holds for $(x \mapsto \mathcal{E}(1-x), \theta)$. A symmetrization of the kind $\tilde{\mathcal{E}}(x) = \frac{1}{2}(\mathcal{E}(x) + \mathcal{E}(1-x))$ will thus not change the right-hand side and at most increase the left-hand side if $\mathcal{E}(\theta) \neq \mathcal{E}(1-\theta)$, making the condition only stricter.

Therefore, an energy function giving the best bound on parameters θ allowing for finally blocked edges through (3.10) can be assumed to be symmetric on $[0,1]$ and having the image set $[0, \frac{1}{2}]$. Set $X := \frac{1}{2} + |\eta_0 - \frac{1}{2}|$, a $[\frac{1}{2}, 1]$-valued random variable, which by the symmetry of η_0 implies $\mathbb{E}[\mathcal{E}(X)] = \mathbb{E}[\mathcal{E}(\eta_0)]$. The largest θ satisfying (3.10) is then the unique one (larger than $\frac{1}{2}$) for which $\mathcal{E}(\theta) = \mathbb{E}[\mathcal{E}(\eta_0)]$. Note that the convexity of the energy function forces it to be strictly monotonous where it is not attaining its minimum, which is 0, and a choice such that $\mathcal{E}(\eta_0) = 0$ a.s. will only give a trivial bound on θ as discussed above.

Another look at Jensen's inequality tells us that $\mathbb{E}[\mathcal{E}(X)] \geq \mathcal{E}(\mathbb{E}\,X)$, with strict inequality if $\mathcal{E}$ is not linear on $\mathrm{supp}(\mathcal{L}(X))$. If this inequality is strict, larger values for θ than $\mathbb{E}\,X$ will also satisfy (3.10). Being linear on $\mathrm{supp}(\mathcal{L}(X))$ and convex means being linear at least on the smallest interval containing the support, i.e. $I := \mathrm{conv}(\mathrm{supp}(\mathcal{L}(X)))$. How $\mathcal{E}$ is defined on $[\frac{1}{2}, 1] \setminus I$ is irrelevant, so we may assume it to be linear on all of $[\frac{1}{2}, 1]$. The assumptions on symmetry and image set finally force $\mathcal{E}$ to be the function $x \mapsto |x - \frac{1}{2}|$.

(d) In the case of an asymmetric distribution of η_0 there are actually better choices.

Consider the example sketched on the right, where $\mathbb{P}(\eta_0 = 0) = \frac{1}{3}$, $\mathbb{P}(\eta_0 = \frac{2}{3}) = \frac{1}{2}$, $\mathbb{P}(\eta_0 = 1) = \frac{1}{6}$, and the energy function is piecewise linear as shown.

Taking $x \mapsto |x - \frac{1}{2}|$ as energy function shows via (3.10) that finally blocked edges are only possible for
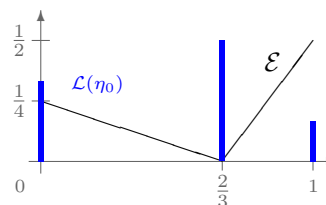
$$\theta \leq \mathbb{E}\left[|\eta_0 - \tfrac{1}{2}|\right] + \tfrac{1}{2} = \tfrac{1}{2}(\tfrac{1}{2} + \tfrac{1}{6}) + \tfrac{1}{2} = \tfrac{5}{6}.$$

Taking $\mathcal{E}$ piecewise linear with $\mathcal{E}(0) = \frac{1}{4}$, $\mathcal{E}(\frac{2}{3}) = 0$ and $\mathcal{E}(1) = \frac{1}{2}$ gives in turn $\mathbb{E}[\mathcal{E}(\eta_0)] = \frac{1}{6} = \mathcal{E}(\frac{2}{9}) = \mathcal{E}(\frac{7}{9})$, hence a.s. no blocked edges for $\theta > \frac{7}{9}$, which is slightly better.

Note however that for every convex $\mathcal{E}$ there are always linear functions l_1, l_2 such that $l_1(1 - \theta) = \mathcal{E}(1 - \theta)$, $l_2(\theta) = \mathcal{E}(\theta)$ and $l_1, l_2 \leq \mathcal{E}$. Taking their maximum will give a convex function leaving the left-hand side of (3.10) unchanged and at most decreasing the right-hand side. By an appropriate affine transformation of the kind $y \mapsto a\,y + c$, $a > 0$ this function can be altered to have image set $[0, \frac{1}{2}]$ without changing the condition on θ that follows from (3.10) as mentioned above. Consequently, the sharpest bound using (3.10) will even in the asymmetric case always be established by some piecewise linear function with only one bend mapping to $[0, \frac{1}{2}]$ as in the example.

(e) It is worth remarking, that the bounds coming from (3.10) applied to the model with i.i.d. initial opinions on $\mathbb{Z}$ are a lot closer to the truth for centered distributions.

The best we can come up with for the uniform case is $\frac{3}{4}$ and for $\mathrm{unif}(\{0, \frac{1}{2}, 1\})$ even $\frac{5}{6}$, whereas Theorem 2.2 tells us that on $\mathbb{Z}$ the actual bound on θ to allow for finally

blocked edges is $\frac{1}{2}$ in either case. In the asymmetric example from above, we get the bound $\theta \leq \frac{7}{9}$ which is not too far off its critical value $\theta_c = \frac{2}{3}$ on $\mathbb{Z}$.

For a distribution of η_0 which is strongly concentrated around the mean, for instance $\mathbb{P}(\eta_0 = 0) = \mathbb{P}(\eta_0 = 1) = \frac{1}{n}$, $\mathbb{P}(\eta_0 = \frac{1}{2}) = \frac{n-2}{n}$, with n large, the bound derived using $x \mapsto |x - \frac{1}{2}|$ as energy function is $\theta \leq \mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] + \frac{1}{2} = \frac{1}{n} + \frac{1}{2}$. The corresponding critical value on $\mathbb{Z}$ according to Theorem 2.2 is again $\frac{1}{2}$, hence quite well approximated.

That we get the right answer for a non-constant distribution concentrated on $\{0, 1\}$ is due to the huge gap. For a slightly changed symmetric version, for example $\mathbb{P}(\eta_0 = 0) = \mathbb{P}(\eta_0 = 1) = \frac{n-1}{2n}$, $\mathbb{P}(\eta_0 = \frac{1}{2}) = \frac{1}{n}$, again n large, however, the best bound we get following the reasoning of the above theorem is

$$\theta \leq \mathbb{E}\left[|\eta_0 - \tfrac{1}{2}|\right] + \tfrac{1}{2} = \tfrac{1}{2} \cdot \tfrac{n-1}{n} + \tfrac{1}{2} = 1 - \tfrac{1}{2n}$$

and this is far off the true value on $\mathbb{Z}$, which is once more $\theta_c = \frac{1}{2}$.

(f) As in Theorem 2.2, the general case where the initial distribution's support is contained in $[a, b]$, $a < b \in \mathbb{R}$, can be treated by appropriate translation and scaling.

In conclusion, the results from Section 2 show that for $d = 1$ and a sequence of initial values satisfying the finite energy condition (see Definition 2.5), there exists a critical parameter θ_c (which is $\frac{1}{2}$ in the standard uniform case) at which a phase transition from no consensus to strong consensus takes place. Strictly weak consensus could only exist for the unsolved case of $\theta = \theta_c$.

Theorem 3.1 states that the case of no consensus is impossible for initial marginal distributions that attribute a positive probability to $(0, 1)$ and θ large enough ($\frac{3}{4}$ in the uniform case).

Remark 3.6. *The results from Theorem 3.1 can actually be generalized from the grid $\mathbb{Z}^d$ to any infinite, locally finite, transitive and amenable (connected) graph $G = (V, E)$. In this generality, the configuration of initial opinions would have to be ergodic with respect to the graph automorphisms instead of shifts, of course.*

Recall that a graph is called locally finite if every vertex has a finite degree, which together with the regularity of a transitive graph implies bounded degree. A graph is called amenable if there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of finite sets such that the ratio of boundary and volume $\frac{|\partial_E F_n|}{|F_n|}$ tends to 0 as $n \to \infty$. Such sequences are called Følner sequences.

In the case of an infinite, locally finite, transitive and amenable connected graph, we can choose the Følner sequence $(F_n)_{n \in \mathbb{N}}$ as an increasing sequence with $\bigcup_{n \in \mathbb{N}} F_n = V$; see the appendix of [6] for further details. As a replacement for Zygmund's ergodic theorem, we can then use the mean ergodic theorem for L^2-functions which can be found as Thm. A.5 in [6], with $(F_n)_{n \in \mathbb{N}}$ stepping in for $(\Lambda_n)_{n \in \mathbb{N}}$:

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{v \in F_n} W_t^{\mathrm{tot}}(v) = \mathbb{E}\left[W_t^{\mathrm{tot}}(\mathbf{0})\right] \quad \text{in } L^2,$$

where $\mathbf{0}$ is some fixed vertex of G. It is not a problem that this result only gives L^2-convergence instead of almost sure convergence, since L^2-convergence is stronger than convergence in probability and the latter implies almost sure convergence of a subsequence, which is enough for our purposes.

3.2 Consequences in terms of stochastic dominance

From the area of probabilistic risk analysis the following orders of stochastic dominance are known, which make it possible to rewrite the results from the foregoing subsection obtained by using energy arguments in a nice way.

Definition 3.7. *Let X, Y be two random variables with finite expectation and $\mathcal{F}_{\mathrm{cx}}$ denote the set of all convex, $\mathcal{F}_{\mathrm{icx}}$ the set of all increasing convex functions on $\mathbb{R}$.*

(i) *X is said to be smaller than Y in the usual stochastic order, commonly denoted by $X \leq_{\mathrm{st}} Y$, if for all $a \in \mathbb{R}$:*

$$\mathbb{P}(X > a) \leq \mathbb{P}(Y > a).$$

(ii) *X is said to be smaller than Y in the convex order, commonly denoted by $X \leq_{\mathrm{cx}} Y$, if for all functions $\varphi \in \mathcal{F}_{\mathrm{cx}}$ for which the corresponding expectations exist:*

$$\mathbb{E}\left[\varphi(X)\right] \leq \mathbb{E}\left[\varphi(Y)\right].$$

(iii) *X is said to be smaller than Y in the increasing convex order, commonly denoted by $X \leq_{\mathrm{icx}} Y$, if for all functions $\varphi \in \mathcal{F}_{\mathrm{icx}}$ for which the corresponding expectations exist:*

$$\mathbb{E}\left[\varphi(X)\right] \leq \mathbb{E}\left[\varphi(Y)\right].$$

It is obvious from the definition that $\leq_{\mathrm{cx}}$ implies $\leq_{\mathrm{icx}}$. Furthermore, the converse is true, if the expectations of both random variables coincide, i.e.

$$X \leq_{\mathrm{cx}} Y \iff X \leq_{\mathrm{icx}} Y \text{ and } \mathbb{E}\, X = \mathbb{E}\, Y,$$

see for example Thm. 4.A.35 in [13].

An easy coupling argument (using quantile transformation) shows that $\leq_{\mathrm{st}}$ implies $\leq_{\mathrm{icx}}$.

Proposition 3.8. *Let $(\eta_t(v))_{t \geq 0}$ denote the piecewise constant jump process describing the value at some fixed vertex $v \in \mathbb{Z}^d$ throughout time, as before. Furthermore, let the initial values again be distributed on $[0, b]$ and $\mathbb{E}\, \eta_0$ be the corresponding expected value.*

For any two points in time $0 \leq s \leq t$, we have $\eta_t(v) \leq_{\mathrm{cx}} \eta_s(v)$. This in turn directly implies $|\eta_t(v) - \mathbb{E}\, \eta_0| \leq_{\mathrm{icx}} |\eta_s(v) - \mathbb{E}\, \eta_0|$.

Proof. First of all, it is worth remarking that the partial orders $\leq_{\mathrm{cx}}$ and $\leq_{\mathrm{icx}}$ are actually defined on the set of distributions and do therefore not depend on a random variable X itself but rather on $\mathcal{L}(X)$. The distribution of $\eta_t(v)$ is by symmetry the same for every $v \in \mathbb{Z}^d$, hence it is enough to consider one fixed vertex.

Let φ be a convex function on $\mathbb{R}$. For every $t \geq 0$ the random variable $\eta_t(v)$ lies in $[0, b]$ and since convexity implies continuity on closed intervals, φ attains its minimum

$$c := \min\left\{\varphi(x),\ x \in [0, b]\right\}.$$

Hence $\mathcal{E} : x \mapsto \varphi(x) - c$ is a non-negative convex function on $[0, b]$ and therefore a proper choice as energy function as outlined in the beginning of the foregoing subsection.

Let $W_t(v) = \mathcal{E}(\eta_t(v))$ denote the energy attributed to the chosen vertex at time t and $W_t^{\mathrm{tot}}(v) = W_t(v) + \frac{1}{2} \sum_{e \in E(v)} W_t^{\mathrm{loss}}(e)$ its total energy, just as in (3.2). Lemma 3.2 tells us that $\mathbb{E}\left[W_t^{\mathrm{tot}}(v)\right] = \mathbb{E}\left[W_0(v)\right]$ for all $t \geq 0$ and the fact that $(W_t^{\mathrm{loss}}(e))_{t \geq 0}$ is non-decreasing

and non-negative for every edge e gives accordingly

$$
\begin{aligned}
\mathbb{E}\left[W_t(v)\right] &= \mathbb{E}\left[W_t^{\mathrm{tot}}(v)\right] - \frac{1}{2} \sum_{e \in E(v)} \mathbb{E}\left[W_t^{\mathrm{loss}}(e)\right] \\
&\leq \mathbb{E}\left[W_s^{\mathrm{tot}}(v)\right] - \frac{1}{2} \sum_{e \in E(v)} \mathbb{E}\left[W_s^{\mathrm{loss}}(e)\right] = \mathbb{E}\left[W_s(v)\right] \\
&\leq \mathbb{E}\left[W_0(v)\right] \quad \text{for } 0 \leq s \leq t.
\end{aligned}
$$

If we plug in the special form of $\mathcal{E}$ chosen above (and add c along the chain of inequalities) this reads:

$$
\mathbb{E}\left[\varphi(\eta_t(v))\right] \leq \mathbb{E}\left[\varphi(\eta_s(v))\right] \quad \left(\ \leq \mathbb{E}\left[\varphi(\eta_0(v))\right]\ \right).
$$

Since $\varphi \in \mathcal{F}_{\mathrm{cx}}$ was arbitrary, this proves the first part of the claim.

To see that $(|\eta_t(v) - \mathbb{E}\,\eta_0|)_{t \geq 0}$ is a non-increasing sequence with respect to $\leq_{\mathrm{icx}}$ one only has to note that the function $x \mapsto |x - \mathbb{E}\,\eta_0|$ is convex. A short moment's thought reveals that the composition of an increasing convex with a convex function is again convex. Thus, for any $\varphi \in \mathcal{F}_{\mathrm{icx}}$ the already proved part applied to the convex function $x \mapsto \varphi(|x - \mathbb{E}\,\eta_0|)$ provides

$$
\mathbb{E}\left[\varphi(|\eta_t(v) - \mathbb{E}\,\eta_0|)\right] \leq \mathbb{E}\left[\varphi(|\eta_s(v) - \mathbb{E}\,\eta_0|)\right],
$$

which in turn proves $|\eta_t(v) - \mathbb{E}\,\eta_0| \leq_{\mathrm{icx}} |\eta_s(v) - \mathbb{E}\,\eta_0|$. □

This proposition in hand makes it possible to reprove the result from Theorem 3.1: Already in 1979, Meilijson and Nádas [12] showed that $Y \leq_{\mathrm{icx}} X$ implies $Y \leq_{\mathrm{st}} h_{\mathcal{L}(X)}(X)$, where the function h_μ denotes the mean residual life of a random variable with distribution μ, i.e.:

$$
\text{For } Z \sim \mu \text{ and } t \in \mathbb{R} \text{ s.t. } \mu\big((t, \infty)\big) > 0 : \ h_\mu(t) := \mathbb{E}\left[Z \mid Z > t\right].
$$

Having the initial distribution $\mathcal{L}(\eta_0) = \mathrm{unif}([0,1])$ means $|\eta_0 - \frac{1}{2}| \sim \mathrm{unif}([0, \frac{1}{2}])$, which gives

$$
h_{\mathrm{unif}([0,1/2])}(t) = \tfrac{1}{4} + \tfrac{t}{2}.
$$

Consequently, we get $|\eta_t - \frac{1}{2}| \leq_{\mathrm{st}} \frac{1}{4} + \frac{Z}{2}$, where $Z \sim \mathrm{unif}([0, \frac{1}{2}])$, another contradiction to (3.9) if $\theta > \frac{3}{4}$.

That the processes $(\eta_t(v))_{t \geq 0}$ are non-increasing in the convex order renders it possible to conclude convergence in distribution. This however is far from the almost sure convergence derived in the one-dimensional case.

Proposition 3.9. *Let $(\eta_t(v))_{t \geq 0}$ be as before. There exists a $[0, b]$-valued random variable η_∞ such that $\eta_t(v) \xrightarrow{d} \eta_\infty$ for every $v \in \mathbb{Z}^d$.*

Proof. Again, symmetry ensures that if the statement holds true for some vertex v it is valid for all such. Building on a famous result of Straßen and following ideas of Doob, Kellerer showed in 1972 that for a family of probability measures $\{\mu_t\}_{t \geq 0}$ which is non-decreasing in the increasing convex order there always exists a submartingale with the corresponding marginals, see Thm. 3 in [9]. Therefore, the non-increasing family $\{\mathcal{L}(\eta_t(v))\}_{t \geq 0}$ can be interpreted as the marginal distributions of a supermartingale $(X_t)_{t \geq 0}$. As the mean of these distributions is constant, which follows from Lemma 3.2 as mentioned in the above remark and corresponds to the stronger condition of non-increasing ordering w.r.t. $\leq_{\mathrm{cx}}$, $(X_t)_{t \geq 0}$ actually is a martingale.

Doob's martingale convergence theorem guarantees a random variable X_∞ such that $(X_t)_{t \geq 0}$ converges to X_∞ almost surely, hence in distribution. Writing η_∞ instead of X_∞ establishes the claim. □

4 On the infinite cluster of supercritical bond percolation

In this section we consider the Deffuant opinion dynamics on the random subgraph of $\mathbb{Z}^d$, $d \geq 2$, which is formed by supercritical i.i.d. bond percolation, independent of the initial configuration and the Poisson processes determining the times of potential opinion updates.

That means, each edge of the grid is independently chosen to be open with a fixed probability $p \in (0, 1]$. One of the classical results in percolation theory tells us that for $d \geq 2$, there exists a critical value $p_c(d) \in (0, 1)$ for p above which we will a.s. find an infinite cluster and that this cluster is a.s. unique. The common notation for the event that some vertex v sits in the infinite cluster is $\{v \leftrightarrow \infty\}$. Slightly abusing this notation we will write $\{e \leftrightarrow \infty\}$ for the event that the edge e is part of the infinite cluster.

The fact that ergodicity, one essential element to derive the results from the foregoing section, is preserved when we consider the (random) subgraph of $\mathbb{Z}^d$ formed by i.i.d. bond percolation allows for an immediate transfer of the corresponding results for the whole grid.

Lemma 4.1. *Let the Deffuant model with initial values drawn from a distribution on $[0, b]$ and parameter $\theta \in (0, b]$ be as above, but now take place on the graph of a super-critical i.i.d. bond percolation on $\mathbb{Z}^d$ which is independent of the initial configuration and the Poisson processes. Then the lemmas of the foregoing section extend as follows:*

(a) $\mathbb{E}\left[W_t^{\text{tot}}(v) \,|\, v \leftrightarrow \infty\right] = \mathbb{E}\left[W_0(\mathbf{0})\right]$

(b) Given the edge $\langle u, v \rangle$ is open, we get as in Lemma 3.3 that a.s.
   $|\eta_t(u) - \eta_t(v)| > \theta$ for sufficiently large t or $\lim_{t \to \infty} |\eta_t(u) - \eta_t(v)| = 0$.

(c) The probability that some edges of the infinite cluster will be finally blocked in the Deffuant model is either 0 or 1.

Proof. (a) Using the notation from Lemma 3.2 and its line of reasoning, it is obvious that the process $\{W_t^{\text{tot}}(v) \cdot \mathbb{1}_{\{v \leftrightarrow \infty\}}\}_{v \in \mathbb{Z}^d}$ is ergodic with respect to shifts. Hence instead of (3.4) one has

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{v \in C_\infty \cap \Lambda_n} W_t^{\text{tot}}(v) = \mathbb{E}\left[W_t^{\text{tot}}(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0} \leftrightarrow \infty\}}\right] \text{ a.s.,} \tag{4.1}$$

where C_∞ denotes the infinite percolation cluster. By the same argument as in the quoted lemma, the left-hand side is constant over time and we thus get

$$
\begin{aligned}
\mathbb{P}(v \leftrightarrow \infty) \cdot \mathbb{E}\left[W_t^{\text{tot}}(v) \,|\, v \leftrightarrow \infty\right] &= \mathbb{E}\left[W_t^{\text{tot}}(v) \cdot \mathbb{1}_{\{v \leftrightarrow \infty\}}\right] \\
&= \mathbb{E}\left[W_t^{\text{tot}}(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0} \leftrightarrow \infty\}}\right] \\
&= \mathbb{E}\left[W_0(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0} \leftrightarrow \infty\}}\right] \\
&= \mathbb{P}(\mathbf{0} \leftrightarrow \infty) \cdot \mathbb{E}\left[W_0(\mathbf{0})\right],
\end{aligned}
$$

using symmetry and independence. Dividing by the probability for percolation of a given vertex $\mathbb{P}(v \leftrightarrow \infty)$, which is non-zero for supercritical percolation, yields the claim.

(b) To get the second statement one simply has to mimick Lemma 3.3. The only things changing are that we have to condition on the event of $e = \langle u, v \rangle$ being open in the realization of the i.i.d. bond percolation and the probability at a given point in time that e will be the next edge incident to either u or v where a Poisson event occurs is no longer precisely $\frac{1}{4d-1}$ but bounded from below by the same value (since some of the other edges might be closed).

(c) Following the proof of Lemma 3.4, let us consider the probability that some given edge e is open, further belongs to the infinite percolation component and is finally blocked in the Deffuant dynamics. If

$$p_{\text{block}} := \mathbb{P}(e \leftrightarrow \infty, e \text{ finally blocked}) = 0,$$

the union bound and part (b) guarantee that a.s. all neighbors in the infinite component will finally concur. If this probability is positive, however, and $N(v)$ denotes the number of edges incident to v, open in the realization of the i.i.d. bond percolation, that will get finally blocked in the Deffuant model, another application of Zygmund's Ergodic Theorem yields:

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{v \in C_\infty \cap \Lambda_n} N(v) = \mathbb{E}\left[N(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0} \leftrightarrow \infty\}}\right] = 2d \cdot p_{\text{block}} > 0 \text{ a.s.}$$

Hence with probability 1, there will be (infinitely many) edges that belong to the infinite percolation component and are finally blocked.

$\square$

Having checked that these auxiliary results transfer appropriately to the setting of supercritical percolation, the following equivalent to Theorem 3.1 can be verified with the very same reasoning as before:

Theorem 4.2. *Consider the Deffuant model on the subgraph of $\mathbb{Z}^d$, $d \geq 2$, formed by an independent supercritical i.i.d. bond percolation as described above.*

(a) *If the initial values are distributed uniformly on $[0,1]$ and $\theta > \frac{3}{4}$, a.s. we will finally have weak consensus in the infinite percolation cluster, i.e. for all $u, v \in \mathbb{Z}^d$ given the event $\{u, v \leftrightarrow \infty\}$ we have*

$$\mathbb{P}\left(\lim_{t \to \infty} |\eta_t(u) - \eta_t(v)| = 0\right) = 1.$$

(b) *For general initial distributions on $[0,1]$, the range of θ, where final consensus of the infinite cluster is guaranteed, is non-trivial, i.e. including values smaller than 1, unless the initial values are concentrated on 0 and 1, taking on both values with positive probability.*

Proof. Given the event that $v \in \mathbb{Z}^d$ is in the infinite percolation cluster which contains (open) edges that are finally blocked by the opinion dynamics we get as in (3.9)

$$\liminf_{t \to \infty} |\eta_t(v) - \tfrac{1}{2}| \geq \theta - \tfrac{1}{2} \text{ a.s.}$$

Choosing again $\mathcal{E} : x \mapsto |x - \frac{1}{2}|$ as energy function the above lemma and the conditional version of Fatou's Lemma yield the following chain of inequalities:

$$\begin{aligned}
\theta - \tfrac{1}{2} &\leq \mathbb{E}\left[\liminf_{t \to \infty} |\eta_t(v) - \tfrac{1}{2}| \mid v \leftrightarrow \infty\right] \\
&\leq \liminf_{t \to \infty} \mathbb{E}\left[|\eta_t(v) - \tfrac{1}{2}| \mid v \leftrightarrow \infty\right] \\
&\leq \liminf_{t \to \infty} \mathbb{E}\left[W_t^{\text{tot}}(v) \mid v \leftrightarrow \infty\right] \\
&= \mathbb{E}\left[W_0^{\text{tot}}(v)\right] = \mathbb{E}\left[|\eta_0(v) - \tfrac{1}{2}|\right].
\end{aligned}$$

Consequently, for blocked edges to occur in the infinite percolation cluster we have to have $\theta \leq \frac{3}{4}$ in the standard case of unif$([0,1])$ initial opinions and $\theta \leq \frac{1}{2} + \mathbb{E}\left[|\eta_0(v) - \frac{1}{2}|\right]$ in the general case.

$\square$

So far, this seems like just a generalization of Section 3. In the percolation setting however, a coupling argument allows to prove a result concerning the other end of the θ-spectrum, under slightly stronger conditions on the initial opinion configuration (see also Remark 4.5 below).

Theorem 4.3. *Consider again the Deffuant model on the infinite cluster of supercritical percolation, this time with i.i.d. initial opinion values distributed on $[0,1]$, s.t. $[0,1]$ is the minimal closed interval containing the support of the marginal distribution. In addition, we require the percolation parameter p to be less than 1.*

For $\theta < \frac{1}{2}$ the probability that the opinion dynamics approach strong consensus on the infinite percolation cluster is 0.

Proof. The line of reasoning to prove this statement is by contradiction. Assuming strong consensus for some fixed value of θ in $(0, \frac{1}{2})$, we are going to show that there will be finally blocked edges in the infinite percolation component with positive probability. This contradicts part (c) of Lemma 4.1.

To that end let us consider two coupled copies of the supercritical i.i.d. bond percolation, see Figure 2. Fix an edge $e = \langle u, v \rangle$ and let the two copies coincide on $E(\mathbb{Z}^d) \setminus \{e\}$. Let $p \in (0, 1)$ denote the probability for an edge to be open in the percolation model and A be the event that the edges incident to u other than e are closed and v sits in the infinite component. By a coupling argument using local modification it can easily be seen that this event has positive probability if p is supercritical.

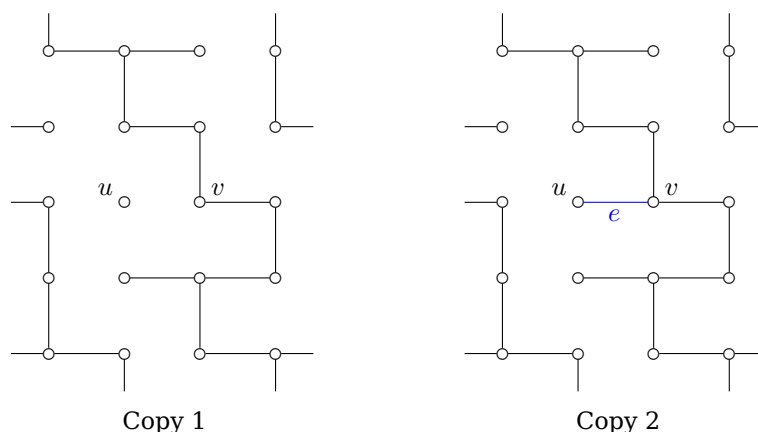Copy 1   Copy 2

Figure 2: Two appropriately coupled copies of the same i.i.d. percolation process on $\mathbb{Z}^d$ on which the opinion dynamics procedure takes place.

Now we want to couple the two copies in such a way that with positive probability e is closed in copy 1 and open in copy 2 under the event A. Let U be a unif($[0,1]$)-distributed random variable, independent of the percolation process on $E(\mathbb{Z}^d) \setminus \{e\}$. Declare e to be open in copy 1 if $U < p$, closed otherwise, and open in copy 2 if $U > 1 - p$ and closed otherwise. This defines two proper i.i.d. bond percolation processes.

If B denotes the event that the edge e is closed in copy 1 and open in copy 2, we get $\mathbb{P}(B) = \min\{p, 1 - p\} > 0$. By independence we also have that the event $A \cap B$ has positive probability.

Since the event that there is strong consensus on the infinite percolation cluster is ergodic with respect to shifts, it is a 0-1-event. Due to the assumption it must have probability 1. Define $\delta := \frac{1}{2} - \theta$, which is positive.

Let us now restrict our attention to the event $A \cap B$ and the first copy. Since v lies

in the infinite component, there is a time $T < \infty$ s.t.

$$\mathbb{P}\big(|\eta_t(v) - \mathbb{E}\,\eta_0| < \tfrac{\delta}{2} \text{ for all } t \geq T \mid A \cap B\big) > 0. \tag{4.2}$$

Note that given $A \cap B$, in copy 1 the process $(\eta_t(v))_{t \geq 0}$ is independent of $\eta_0(u)$ as well as the Poisson process attributed to e. By the choice of θ and the properties of the initial distribution we get in addition:

$$\mathbb{P}\big(\eta_0(u) \notin [\mathbb{E}\,\eta_0 - (\theta + \tfrac{\delta}{2}), \mathbb{E}\,\eta_0 + (\theta + \tfrac{\delta}{2})]\big) > 0.$$

If we finally define C to be the event that $A \cap B$ occurs, no Poission event occurs at e before T, $|\eta_t(v) - \mathbb{E}\,\eta_0| < \tfrac{\delta}{2}$ for all $t \geq T$ and $|\eta_0(u) - \mathbb{E}\,\eta_0| \geq \theta + \tfrac{\delta}{2}$, independence of the latter events conditioned on $A \cap B$ makes sure that C occurs with positive probability.

If we run the opinion dynamics on both copies simultaneously it is obvious that they behave identically as long as no Poisson event occurs for e. Given the event C the values at u and v are further than θ apart from time T on. Hence, even in the second copy, there will be no interaction between the two since no Poisson event occurs at e before time T. In other words, with probability at least $\mathbb{P}(C) > 0$ there will be no consensus in the infinite percolation cluster of the second copy, to which given $A \cap B$ both u and v belong. Since both copies underly the same distribution, this contradicts the assumption that we have strong consensus. It is worth noting that strictly weak consensus can not be excluded since the argument in (4.2) does not hold for the weak case. $\qquad\square$

Remark 4.4. *The two results of Theorem 4.2 and 4.3 put together imply the following: The Deffuant model on the infinite cluster, formed by supercritical i.i.d. bond percolation on $\mathbb{Z}^d$ with non-trivial percolation parameter $p \in (p_c, 1)$, featuring i.i.d. initial opinions having a non-degenerate marginal distribution on $[0, 1]$ – in the sense that it attributes positive probability to $(0, 1)$, $[0, \varepsilon)$ and $(1 - \varepsilon, 1]$ for all $\varepsilon > 0$ – either approaches weak consensus for all $\theta \in (0, 1]$ or there is a phase transition in this parameter.*

Remark 4.5. *Similarly to the ideas in Subsection 2.2, we can relax the strong condition of independence when it comes to the initial opinion values and still receive the same result. In the proof of Theorem 4.3, the only instance where more than stationarity and ergodicity with respect to shifts of the initial configuration $\{\eta_0(v)\}_{v \in \mathbb{Z}^d}$ was used is in the conclusion that the event C has positive probability. This however can also be guaranteed without the independence of initial opinion values, if only $\{\eta_0(v)\}_{v \in \mathbb{Z}^d}$ additionally satisfies the finite energy condition as laid down in Definition 2.5 but now with $\mathbb{Z}^d$ in place of $\mathbb{Z}$.*

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