COVARIANCE STRUCTURE OF PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH MULTIPLICATIVE LÉVY NOISE

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ABSTRACT. We consider parabolic stochastic partial differential equations driven by multiplicative Lévy noise of an affine type. For the second moment of the mild solution, we derive a well-posed deterministic space-time variational problem posed on tensor product spaces, which subsequently leads to a deterministic equation for the covariance function.

1. INTRODUCTION

The covariance function of a stochastic process provides information about the correlation of the process with itself at pairs of time points and, hence, about the strength of their linear relation. In addition, it shows if this behavior is stationary, i.e., whether or not it changes when shifted in time and if it follows any trends. Since the covariance of the solution process to a parabolic stochastic partial differential equation driven by an additive Q-Wiener process has been described as the solution to a deterministic, tensorized evolution equation in [7], the immediate question has arisen, if it is possible to establish such a correspondence also for covariance functions of solutions to stochastic partial differential equations driven by multiplicative noise.

In this paper, we extend the investigation of the covariance function to solution processes of parabolic stochastic partial differential equations driven by multiplicative Lévy noise considered, e.g., in [9]. The multiplicative operator is assumed to be affine-linear. Clearly, under appropriate assumptions on the driving Lévy process, the mean function of the mild solution satisfies the corresponding deterministic, parabolic evolution equation as in the case of additive Wiener noise, since in both cases the stochastic integral is a martingale. However, the presence of a multiplicative term changes the behavior of the second moment and the covariance. We prove that also in this case the second moment as well as the covariance of the square-integrable mild solution satisfy deterministic space-time variational problems posed on tensor products of Bochner spaces. In contrast to the case of additive Wiener noise considered in [7], the resulting bilinear form does not arise from taking the tensor of the corresponding deterministic parabolic operator with

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itself, but it involves a non-separable operator in the dual space. Because of this term, well-posedness of the derived deterministic variational problems is not an immediate consequence.

The present paper is structured as follows: In Section 2 we present the parabolic stochastic differential equation, whose covariance function we aim to describe, as well as some auxiliary results. More precisely, in Section 2.1 we first define the mild solution of the investigated stochastic partial differential equation. In addition, we impose appropriate assumptions on the initial value and on the affine operator in the multiplicative term such that existence and uniqueness of the mild solution are guaranteed. In Section 2.2 we state and establish some results which we need in order to prove the main theorems of this paper in Sections 3 and 5: the deterministic space-time variational problems satisfied by the second moment and the covariance of the mild solution. We derive a series expansion for the Lévy process as well as an isometry for the weak stochastic integral.

In Theorem 3.5 of Section 3 we show that the second moment of the mild solution solves a deterministic space-time variational problem posed on tensor products of Bochner spaces. In order to be able to formulate this variational problem, we need some additional regularity of the second moment which we prove first.

The aim of Section 4 is to establish well-posedness of the derived variational problem. According to the Nečas theorem, this is equivalent to showing that the inf-sup constant of the arising bilinear form is strictly positive and that a certain surjectivity condition is satisfied. Therefore, in Section 4.1 we investigate the inf-sup constant using the knowledge about the positivity of the inf-sup constant for space-time variational problems considered in [13]. Theorem 4.8 of Section 4.2 provides surjectivity and well-posedness is deduced.

Finally, in Section 5 we use the results of the previous sections to obtain a well-posed space-time variational problem satisfied by the covariance function of the mild solution.

2. Definitions and preliminaries

In this section the investigated stochastic partial differential equation as well as the setting, that we impose on it, are presented. In addition, we prove some auxiliary results that will be needed later on, in Section 3 and 5, respectively, to derive the deterministic equations for the second moment and the covariance of the solution process to the stochastic partial differential equation.

2.1. **The stochastic partial differential equation.**

For two separable Hilbert spaces H_1 and H_2 we denote by $\mathcal{L}(H_1; H_2)$ the space of bounded linear operators mapping from H_1 to H_2. In addition, we write $\mathcal{L}_p(H_1; H_2)$ for the space of the Schatten class operators of p-th order. Here, for $1 \le p < \infty$ an operator $T \in \mathcal{L}(H_1; H_2)$ is called a *Schatten-class operator of p-th order*, if T has a finite p-th Schatten norm, i.e.,

$$\|T\|_{\mathcal{L}_p(H_1;H_2)} := \left(\sum_{n\in\mathbb{N}} s_n(T)^p \right)^{\frac{1}{p}} < +\infty,$$

where $s_1(T) \ge s_2(T) \ge \ldots \ge s_n(T) \ge \ldots \ge 0$ are the singular values of T, i.e., the eigenvalues of the operator $(T^*T)^{1/2}$. Here, $T^* \in \mathcal{L}(H_2; H_1)$ denotes the adjoint of T. If $H_1 = H_2 = H$ we abbreviate $\mathcal{L}_p(H; H)$ by $\mathcal{L}_p(H)$. For the case $p = 1$

and a separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$ and orthonormal basis $(e_n)_{n\in\mathbb{N}}$ we may introduce the *trace* of an operator $T \in \mathcal{L}_1(H)$ by

$$\operatorname{tr}(T) := \sum_{n\in\mathbb{N}} \langle T e_n, e_n \rangle_H.$$

The trace $\operatorname{tr}(T)$ is independent of the choice of the orthonormal basis and it satisfies $|\operatorname{tr}(T)| \le \|T\|_{\mathcal{L}_1(H)}$, cf. [3, Proposition C.1]. By $\mathcal{L}_1^+(H)$ we denote the space of all nonnegative, symmetric trace class operators on H, i.e.,

$$\mathcal{L}_1^+(H) := \{T \in \mathcal{L}_1(H) : \langle T\varphi, \varphi \rangle_H \ge 0, \langle T\varphi, \psi \rangle_H = \langle \varphi, T\psi \rangle_H \quad \forall \varphi, \psi \in H\}.$$

In the following U and H denote separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_H$, respectively.

Let $L := (L(t), t \ge 0)$ be an adapted square-integrable U-valued Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$. More precisely, we assume that

(i) L has independent increments, i.e., for all $0 \le t_0 < t_1 < \ldots < t_n$ the U-valued random variables $L(t_1) - L(t_0)$, $L(t_2) - L(t_1)$, …, $L(t_n) - L(t_{n-1})$ are independent;
(ii) L has stationary increments, i.e., the distribution of $L(t) - L(s)$, $s \le t$, depends only on the difference $t - s$;
(iii) $L(0) = 0$ $\mathbb{P}$-almost surely;
(iv) L is stochastically continuous, i.e.,

$$\lim_{\substack{s\to t\\ s\ge 0}} \mathbb{P}(\|L(t) - L(s)\|_U > \epsilon) = 0 \quad \forall \epsilon > 0, \quad \forall t \ge 0.$$

(v) L is adapted, i.e., $L(t)$ is $\mathcal{F}_t$-measurable for all $t \ge 0$.
(vi) L is square-integrable, i.e., $\mathbb{E}\|L(t)\|_U^2 < +\infty$ for all $t \ge 0$.

Furthermore, we assume that for $t > s \ge 0$ the increment $L(t) - L(s)$ is independent of $\mathcal{F}_s$ and that L has zero mean and covariance operator $Q \in \mathcal{L}_1^+(U)$, i.e., for all $s, t \ge 0$ and $x, y \in U$ it holds: $\mathbb{E}\langle L(t), x \rangle_U = 0$ and

$$\mathbb{E}\left[\langle L(s), x \rangle_U \langle L(t), y \rangle_U\right] = \min\{s, t\}\, \langle Qx, y \rangle_U, \tag{2.1}$$

cf. [9, Theorem 4.44]. Note that under these assumptions, the Lévy process L is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\ge0}$ by [9, Proposition 3.25].

In addition, since $Q \in \mathcal{L}_1^+(U)$ is a nonnegative, symmetric trace class operator, there exists an orthonormal eigenbasis $(e_n)_{n\in\mathbb{N}} \subset U$ of Q with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{\ge0}$, i.e., $Qe_n = \gamma_n e_n$ for all $n \in \mathbb{N}$, and for $x \in U$ we may define the fractional operator $Q^{1/2}$ by

$$Q^{\frac12} x := \sum_{n\in\mathbb{N}} \gamma_n^{\frac12} \langle x, e_n \rangle_U\, e_n,$$

as well as its pseudo inverse $Q^{-1/2}$ by

$$Q^{-\frac12} x := \sum_{n\in\mathbb{N}\,:\,\gamma_n\neq0} \gamma_n^{-\frac12} \langle x, e_n \rangle_U\, e_n.$$

We introduce the vector space $\mathcal{H} := Q^{1/2}U$. Then $\mathcal{H}$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} := \langle Q^{-1/2}\cdot, Q^{-1/2}\cdot \rangle_U$.

Furthermore, let $A\colon \mathcal{D}(A) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator, which is not necessarily bounded, but which has a compact inverse. In this case $-A$ is the generator of an analytic semigroup of contractions $S = (S(t), t \geq 0)$ and for $r \geq 0$ the fractional power operator $A^{r/2}$ is well-defined on a domain $\mathcal{D}(A^{r/2}) \subset H$, cf. [8, Chapter 2]. We define the Hilbert space $\dot{H}^r$ as the completion of $\mathcal{D}(A^{r/2})$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\dot{H}^r} := \langle A^{r/2}\varphi, A^{r/2}\psi \rangle_H.$$

We obtain a scale of Hilbert spaces with $\dot{H}^s \subset \dot{H}^r \subset \dot{H}^0 = H$ for $0 \leq r \leq s$. Its role is to measure spatial regularity. We denote the special case when $r = 1$ by $V := \dot{H}^1$. In this way we obtain a Gelfand triple

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*,$$

where H^* and V^* denote the dual spaces of H and V, respectively. In addition, although the operator A is assumed to be self-adjoint, we denote by $A^* \colon V \to V^*$ its adjoint to clarify whenever we consider the adjoint instead of the operator itself. With these definitions, the operator A and its adjoint are bounded, $A, A^* \in \mathcal{L}(V; V^*)$, since for $\varphi, \psi \in V$ it holds

$${}_{V^*}\langle A\varphi, \psi \rangle_V = \langle A^{1/2}\varphi, A^{1/2}\psi \rangle_H = \langle \varphi, \psi \rangle_V = {}_V\langle \varphi, A^*\psi \rangle_{V^*},$$

where ${}_{V^*}\langle \cdot, \cdot \rangle_V$ and ${}_V\langle \cdot, \cdot \rangle_{V^*}$ denote dual pairings between V and V^*.

We are investigating the stochastic partial differential equation

$$\begin{aligned} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t &= G(X(t))\,\mathrm{d}L(t), \quad t \in \mathbb{T} := [0, T], \\ X(0) &= X_0, \end{aligned} \tag{2.2}$$

for finite $T > 0$. In order to obtain existence and uniqueness of a solution to this problem as well as additional regularity for its second moment, which will be needed later on, we impose the following assumptions on the initial value X_0 and the operator G.

Assumption 2.1. X_0 and G in (2.2) satisfy:

(1) $X_0 \in L^2(\Omega; H)$ is an $\mathcal{F}_0$-measurable random variable.

(2) $G \colon H \to \mathcal{L}_2(\mathcal{H}; H)$ is an affine operator, i.e., $G(\varphi) = G_1(\varphi) + G_2$ with operators $G_1 \in \mathcal{L}(H, \mathcal{L}_2(\mathcal{H}; H))$ and $G_2 \in \mathcal{L}_2(\mathcal{H}; H)$.

(3) There exists a regularity exponent $r \in [0, 1]$ such that $X_0 \in L^2(\Omega; \dot{H}^r)$ and $A^{r/2}S(\cdot)G_1 \in L^2(\mathbb{T}; \mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H)))$, i.e.,

$$\int_0^T \|A^{\frac{r}{2}} S(t) G_1\|^2_{\mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H))}\,\mathrm{d}t < +\infty.$$

(4) $A^{1/2}S(\cdot)G_1 \in L^2(\mathbb{T}; \mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H)))$, i.e.,

$$\int_0^T \|A^{\frac{1}{2}} S(t) G_1\|^2_{\mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H))}\,\mathrm{d}t < +\infty,$$

with the same value for $r \in [0, 1]$ as in (3).

Note that the assumption on G_1 in part (4) implies the one in part (3). Conditions (1)–(3) guarantee $\dot{H}^r$ regularity for the mild solution (cf. Theorem 2.3), but we need all four assumptions for our main results in Sections 3 and 5.

Before we derive the deterministic variational problems satisfied by the second moment and the covariance of the solution X to (2.2) in Sections 3 and 5, we have to specify which kind of solvability we consider. In addition, existence and uniqueness of this solution must be guaranteed.

Definition 2.2. A predictable process $X\colon \Omega\times\mathbb{T}\to H$ is called a mild solution to (2.2), if $\sup_{t\in\mathbb{T}}\|X(t)\|_{L^2(\Omega;H)} < +\infty$ and

$$X(t) = S(t)X_0 + \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s), \quad t\in\mathbb{T}. \tag{2.3}$$

It is a well-known result that there exists a unique mild solution for affine-linear multiplicative noise of the form considered above. More precisely, we have the following theorem.

Theorem 2.3. *Under the conditions (1)–(2) in Assumption 2.1 there exists (up to modification) a unique mild solution X of* (2.2)*. If additionally condition (3) of Assumption 2.1 holds, then the mild solution satisfies*

$$\sup_{t\in\mathbb{T}}\|X(t)\|_{L^2(\Omega;\dot{H}^r)} < +\infty.$$

Proof. The first part of the theorem follows from [9, Theorem 9.29]. Suppose now that condition (3) is satisfied. By the dominated convergence theorem the sequence of integrals

$$\int_0^T \|A^{\frac{r}{2}}S(\tau)G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))}\,\chi_{(0,T/n)}(\tau)\,\mathrm{d}\tau,$$

where $\chi_{(0,T/n)}$ denotes the indicator function on the interval $(0,T/n)$, converges to zero as $n\to\infty$. Therefore, there exists $\widetilde{T}\in(0,T]$ such that

$$\kappa^2 := \int_0^{\widetilde{T}} \|A^{\frac{r}{2}}S(\tau)G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))}\,\mathrm{d}\tau < 1.$$

Define $\widetilde{\mathbb{T}} := \big[0,\widetilde{T}\big]$, $\mathcal{Z} := L^\infty\big(\widetilde{\mathbb{T}};L^2(\Omega;\dot{H}^r)\big)$ and

$$\Upsilon\colon \mathcal{Z}\to\mathcal{Z}, \quad \Upsilon(Z) := S(t)X_0 + \int_0^t S(t-s)G(Z(s))\,\mathrm{d}L(s).$$

Then Υ is a contraction: For every $t\in\widetilde{\mathbb{T}}$ and $Z_1, Z_2\in\mathcal{Z}$ we have

$$\begin{aligned}\|\Upsilon(Z_1)(t)-\Upsilon(Z_2)(t)\|^2_{L^2(\Omega;\dot{H}^r)} &= \mathbb{E}\Big\|\int_0^t S(t-s)G_1(Z_1(s)-Z_2(s))\,\mathrm{d}L(s)\Big\|^2_{\dot{H}^r}\\ &= \mathbb{E}\Big\|\int_0^t A^{\frac{r}{2}}S(t-s)G_1(Z_1(s)-Z_2(s))\,\mathrm{d}L(s)\Big\|^2_{H},\end{aligned}$$

since A and, hence, $A^{r/2}$ are closed operators. Now applying Itô's isometry for the case of a Lévy process, cf. [9, Corollary 8.17], yields

$$\begin{aligned}&= \mathbb{E}\int_0^t \|A^{\frac{r}{2}}S(t-s)G_1(Z_1(s)-Z_2(s))\|^2_{\mathcal{L}_2(\mathcal{H};H)}\,\mathrm{d}s\\ &\le \mathbb{E}\int_0^t \|A^{\frac{r}{2}}S(t-s)G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))}\|Z_1(s)-Z_2(s)\|^2_{\dot{H}^r}\,\mathrm{d}s\\ &\le \sup_{s\in\widetilde{\mathbb{T}}}\mathbb{E}\|Z_1(s)-Z_2(s)\|^2_{\dot{H}^r}\int_0^{\widetilde{T}} \|A^{\frac{r}{2}}S(\tau)G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))}\,\mathrm{d}\tau\\ &\le \kappa^2\|Z_1-Z_2\|^2_{\mathcal{Z}}.\end{aligned}$$

Therefore, $\|\Upsilon(Z_1) - \Upsilon(Z_2)\|_{\mathcal{Z}} \leq \kappa \|Z_1 - Z_2\|_{\mathcal{Z}}$, and Υ is a contraction. By the Banach fixed point theorem, there exists a unique fixed point X_* of Υ in $\mathcal{Z}$. Hence, $X = X_*$ is the unique mild solution to (2.2) on $\widetilde{\mathbb{T}}$ and

$$\|X\|_{\mathcal{Z}}^2 = \sup_{t\in\widetilde{\mathbb{T}}} \mathbb{E}\|X(t)\|_{\dot{H}^r}^2 < +\infty.$$

The claim of the theorem follows from iterating the same argument on the intervals

$$\big[(m-1)\widetilde{T}, \min\{m\widetilde{T}, T\}\big], \quad m \in \big\{1, 2, \ldots, \lceil T/\widetilde{T}\rceil\big\}. \qquad \square$$

2.2. **Auxiliary results.** The aim of this part is to prove some auxiliary results that will be needed later on to derive the main results for the second moment and the covariance of the mild solution X in (2.3). We start with two general results on Lévy processes: First, we introduce a series expansion of the Lévy process in Lemma 2.4, similarly to the Karhunen–Loève expansion for a Wiener process, cf. [3, Proposition 4.1]. Afterwards, we use this expansion to deduce an isometry for the weak stochastic integral in Lemma 2.5, analogously to [7, Lemma 3.1]. This isometry together with the equality stated in Lemma 2.7 will be essential for the derivation of the deterministic equations, whereas Lemma 2.6 will play a crucial role for proving their well-posedness.

Lemma 2.4. *For all $t \in \mathbb{T}$, $L(t)$ admits the expansion*

$$L(t) = \sum_{n\in\mathbb{N}} \sqrt{\gamma_n}\, L_n(t) e_n, \tag{2.4}$$

where $(e_n)_{n\in\mathbb{N}} \subset U$ is an orthonormal eigenbasis of Q with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ is a series of mutually uncorrelated real-valued Lévy processes,

$$L_n(t) = \begin{cases} \gamma_n^{-\frac{1}{2}} \langle L(t), e_n\rangle_U, & \text{if } \gamma_n > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

The series in (2.4) converges in $L^2(\Omega; L^\infty(\mathbb{T}; U))$.

Proof. The proof is adapted from the case of a Wiener process, cf. [3, Proposition 4.1 and Theorem 4.3].

The real-valued processes $(L_n)_{n\in\mathbb{N}}$ defined in (2.5) are Lévy processes since they satisfy the properties (i)–(iv) stated in the beginning of Section 2.1. To see that they are mutually uncorrelated, let $0 \leq s \leq t \leq T$. Then the definition (2.1) of the operator Q yields

$$\begin{aligned} \mathbb{E}\left[L_n(t) L_m(s)\right] &= \frac{1}{\sqrt{\gamma_n \gamma_m}} \mathbb{E}\left[\langle L(t), e_n\rangle_U \langle L(s), e_m\rangle_U\right] \\ &= \frac{s}{\sqrt{\gamma_n\gamma_m}} \langle Q e_n, e_m\rangle_U = \frac{s}{\sqrt{\gamma_n\gamma_m}} \gamma_n \delta_{nm} = s\, \delta_{nm}, \end{aligned}$$

where δ_{nm} denotes the Kronecker delta. In order to prove convergence of the series in (2.4), let $N, M \in \mathbb{N}$, $M < N$. Then, by Parseval's identity,

$$\begin{aligned} \Big\| \sum_{n=M+1}^{N} \sqrt{\gamma_n} L_n(t) e_n \Big\|_{L^2(\Omega; L^\infty(\mathbb{T};U))}^2 &= \mathbb{E} \sup_{t\in\mathbb{T}} \Big\| \sum_{n=M+1}^{N} \sqrt{\gamma_n} L_n(t) e_n \Big\|_U^2 \\ &= \mathbb{E} \sup_{t\in\mathbb{T}} \sum_{n=M+1}^{N} \gamma_n L_n(t)^2 \leq \sum_{n=M+1}^{N} \gamma_n\, \mathbb{E}\left[\sup_{t\in\mathbb{T}} L_n(t)^2\right]. \end{aligned}$$

Since the processes $(L_n)_{n\in\mathbb{N}}$ are right-continuous martingales, we may apply Doob's L^p-inequality for $p=2$, cf. [10, Theorem II.(1.7)], and obtain

$$\Big\| \sum_{n=M+1}^{N} \sqrt{\gamma_n} L_n(t) e_n \Big\|^2_{L^2(\Omega;L^\infty(\mathbb{T};U))} \le \sum_{n=M+1}^{N} \gamma_n \left(\frac{2}{2-1}\right)^2 \sup_{t\in\mathbb{T}} \mathbb{E}\left[L_n(t)^2\right]$$
$$= 4 \sum_{n=M+1}^{N} \gamma_n T \le 4T \ \mathrm{tr}(Q).$$

Hence, the sequence of the partial sums is a Cauchy sequence in $L^2(\Omega; L^\infty(\mathbb{T};U))$. □

Before we formulate the next result, we have to introduce some notation: By $C^0(\mathbb{T};H)$ we denote the space of continuous mappings from $\mathbb{T}=[0,T]$ to the Hilbert space H. Besides, we consider the space $L^2(\Omega\times\mathbb{T};\mathcal{L}_2(\mathcal{H};H))$ of square-integrable $\mathcal{L}_2(\mathcal{H};H)$-valued functions with respect to the measure space $(\Omega\times\mathbb{T},\mathcal{P}_\mathbb{T},\mathbb{P}\otimes\lambda)$, where $\mathcal{P}_\mathbb{T}$ denotes the σ-algebra of predictable subsets of $\Omega\times\mathbb{T}$ and λ the Lebesgue measure on $\mathbb{T}$.

In addition to these function spaces, we will need the notion of tensor product spaces. For two separable Hilbert spaces H_1 and H_2 we define the tensor space $H_1\otimes H_2$ as the completion of the algebraic tensor product between H_1 and H_2 with respect to the norm induced by the following inner product

$$\langle \vartheta\otimes\varphi, \chi\otimes\psi\rangle_{H_1\otimes H_2} := \langle\vartheta,\chi\rangle_{H_1}\langle\varphi,\psi\rangle_{H_2}, \quad \vartheta,\chi\in H_1, \quad \varphi,\psi\in H_2.$$

If $H_1=H_2=H$, we abbreviate $H^{(2)}:=H\otimes H$. Furthermore, for a U-valued Lévy process L with covariance operator Q as considered above, we define the covariance kernel $q\in U^{(2)}$ as the unique element in the tensor space $U^{(2)}$ satisfying

$$\langle q, x\otimes y\rangle_{U^{(2)}} = \langle Qx, y\rangle_U \tag{2.6}$$

for all $x,y\in U$. Note that for an orthonormal eigenbasis $(e_n)_{n\in\mathbb{N}}\subset U$ of Q with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}}$ we may expand q as follows,

$$q = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \langle q, e_n\otimes e_m\rangle_{U^{(2)}}(e_n\otimes e_m) = \sum_{n\in\mathbb{N}} \gamma_n (e_n\otimes e_n), \tag{2.7}$$

since $(e_n\otimes e_m)_{n,m\in\mathbb{N}}$ is an orthonormal basis of $U^{(2)}$ and $\langle q, e_n\otimes e_m\rangle_{U^{(2)}} = \gamma_n\delta_{nm}$.

Lemma 2.5. *For $v_1,v_2\in C^0(\mathbb{T};H)$, a predictable process $\Phi\in L^2(\Omega\times\mathbb{T};\mathcal{L}_2(\mathcal{H};H))$ and $t\in\mathbb{T}$, the weak stochastic integral, cf.* [9, p. 151], *satisfies*

$$\mathbb{E}\Big[\int_0^t \langle v_1(s), \Phi(s)\,\mathrm{d}L(s)\rangle_H \int_0^t \langle v_2(r), \Phi(r)\,\mathrm{d}L(r)\rangle_H\Big]$$
$$= \int_0^t \langle v_1(s)\otimes v_2(s), \mathbb{E}[\Phi(s)\otimes\Phi(s)]q\rangle_{H^{(2)}}\,\mathrm{d}s,$$

where $q\in U^{(2)}$ has been defined in (2.6).

Proof. For $t\in\mathbb{T}$ and $\ell\in\{1,2\}$, the series expansion (2.4) and the properties of the weak stochastic integral yield

$$\int_0^t \langle v_\ell(s), \Phi(s)\,\mathrm{d}L(s)\rangle_H = \sum_{n\in\mathbb{N}} \sqrt{\gamma_n} \int_0^t \langle v_\ell(s), \Phi(s) e_n\rangle_H \,\mathrm{d}L_n(s), \tag{2.8}$$

where this equality holds in $L^2(\Omega;\mathbb{R})$. In addition, since the predictability of Φ implies the predictability of the integrands, we may apply Itô's isometry for real-valued Lévy processes, cf. [2, Theorem 4.2.3], which, along with the polarisation identity, yields

$$\mathbb{E}\Big[\int_0^t \langle v_1(s), \Phi(s)e_n\rangle_H \, \mathrm{d}L_n(s) \int_0^t \langle v_2(r), \Phi(r)e_n\rangle_H \, \mathrm{d}L_n(r)\Big] \\ = \mathbb{E}\left[\int_0^t \langle v_1(s), \Phi(s)e_n\rangle_H \langle v_2(s), \Phi(s)e_n\rangle_H \, \mathrm{d}s\right].$$

We use this equality together with the series expansion (2.8) and the mutual uncorrelation of the processes $(L_n)_{n\in\mathbb{N}}$, which implies the mutual uncorrelation of the corresponding stochastic integrals. This yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \langle v_1(s), \Phi(s)\,\mathrm{d}L(s)\rangle_H \int_0^t \langle v_2(r), \Phi(r)\,\mathrm{d}L(r)\rangle_H\right] \\
&= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \sqrt{\gamma_n\gamma_m}\, \mathbb{E}\left[\int_0^t \langle v_1(s), \Phi(s)e_n\rangle_H \, \mathrm{d}L_n(s) \int_0^t \langle v_2(r), \Phi(r)e_m\rangle_H \, \mathrm{d}L_m(r)\right] \\
&= \sum_{n\in\mathbb{N}} \gamma_n\, \mathbb{E}\left[\int_0^t \langle v_1(s), \Phi(s)e_n\rangle_H \, \mathrm{d}L_n(s) \int_0^t \langle v_2(r), \Phi(r)e_n\rangle_H \, \mathrm{d}L_n(r)\right] \\
&= \sum_{n\in\mathbb{N}} \gamma_n\, \mathbb{E}\left[\int_0^t \langle v_1(s), \Phi(s)e_n\rangle_H \langle v_2(s), \Phi(s)e_n\rangle_H \, \mathrm{d}s\right] \\
&= \sum_{n\in\mathbb{N}} \gamma_n\, \mathbb{E}\left[\int_0^t \langle v_1(s)\otimes v_2(s), \Phi(s)e_n \otimes \Phi(s)e_n\rangle_{H^{(2)}} \, \mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^t \Big\langle v_1(s)\otimes v_2(s), \sum_{n\in\mathbb{N}} \gamma_n \left(\Phi(s)e_n \otimes \Phi(s)e_n\right)\Big\rangle_{H^{(2)}} \, \mathrm{d}s\right] \\
&= \int_0^t \Big\langle v_1(s)\otimes v_2(s), \mathbb{E}\left[\Phi(s)\otimes\Phi(s)\right]\Big[\sum_{n\in\mathbb{N}} \gamma_n (e_n\otimes e_n)\Big]\Big\rangle_{H^{(2)}} \, \mathrm{d}s
\end{aligned}$$

and because of (2.7) we obtain

$$= \int_0^t \langle v_1(s)\otimes v_2(s), \mathbb{E}[\Phi(s)\otimes\Phi(s)]q\rangle_{H^{(2)}} \, \mathrm{d}s. \qquad \square$$

The bilinear form and the right-hand side, which will appear in the variational problem for the second moment, contain several terms depending on the operators G_1 and G_2 as well as on the kernel q that is associated with the covariance operator Q via (2.6). To verify that they are well-defined we will need the following lemma.

Lemma 2.6. *Let $Q \in \mathcal{L}_1^+(U)$ and define $q \in U^{(2)}$ as in (2.6). Then for an affine operator G satisfying condition (2) of Assumption 2.1 the following statements hold:*

(i) *The linear operator $(G_1\otimes G_1)(\cdot)q \colon V^{(2)} \to H^{(2)}$ is bounded and*

$$\|(G_1\otimes G_1)(\cdot)q\|_{\mathcal{L}(V^{(2)};H^{(2)})} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}.$$

(ii) *The linear operators* $(G_1(\cdot)\otimes G_2)q$ *and* $(G_2\otimes G_1(\cdot))q\colon V\to H^{(2)}$ *are bounded and*

$$\begin{aligned}\|(G_1(\cdot)\otimes G_2)q\|_{\mathcal{L}(V;H^{(2)})} &= \|(G_2\otimes G_1(\cdot))q\|_{\mathcal{L}(V;H^{(2)})}\\ &\le \|G_1\|_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}\|G_2\|_{\mathcal{L}_4(\mathcal{H};H)}.\end{aligned}$$

(iii) $(G_2\otimes G_2)q\in H^{(2)}$ *and* $\|(G_2\otimes G_2)q\|_{H^{(2)}}=\|G_2\|^2_{\mathcal{L}_4(\mathcal{H};H)}$.

Proof. Let $(e_n)_{n\in\mathbb{N}}\subset U$ be an orthonormal eigenbasis of the covariance operator $Q\in\mathcal{L}_1^+(U)$ with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}}$ and define

$$\begin{aligned}f_n &:= \sqrt{\gamma_n}e_n, \quad n\in\mathbb{N},\\ \mathcal{I} &:= \{j\in\mathbb{N}:\gamma_j\neq 0\}.\end{aligned}$$

We may expand $q\in U^{(2)}$ as in (2.7). Therefore,

$$q=\sum_{n\in\mathbb{N}}\gamma_n(e_n\otimes e_n)=\sum_{j\in\mathcal{I}}(f_j\otimes f_j).$$

Assume first that $\psi\in V^{(2)}$ with $\psi=\psi_1\otimes\psi_2$ for $\psi_1,\psi_2\in V$. For this case we calculate for (i)

$$\begin{aligned}\|(G_1\otimes G_1)(\psi)q\|^2_{H^{(2)}} &= \Big\|(G_1\otimes G_1)(\psi_1\otimes\psi_2)\sum_{j\in\mathcal{I}}(f_j\otimes f_j)\Big\|^2_{H^{(2)}}\\
&= \Big\langle\sum_{j\in\mathcal{I}}(G_1\otimes G_1)(\psi_1\otimes\psi_2)(f_j\otimes f_j),\sum_{k\in\mathcal{I}}(G_1\otimes G_1)(\psi_1\otimes\psi_2)(f_k\otimes f_k)\Big\rangle_{H^{(2)}}\\
&= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}}\langle G_1(\psi_1)f_j\otimes G_1(\psi_2)f_j, G_1(\psi_1)f_k\otimes G_1(\psi_2)f_k\rangle_{H^{(2)}}\\
&= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}}\langle G_1(\psi_1)f_j, G_1(\psi_1)f_k\rangle_H\langle G_1(\psi_2)f_j, G_1(\psi_2)f_k\rangle_H\\
&= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}}\langle G_1(\psi_1)^*G_1(\psi_1)f_j, f_k\rangle_{\mathcal{H}}\langle G_1(\psi_2)^*G_1(\psi_2)f_j, f_k\rangle_{\mathcal{H}},\end{aligned}$$

where $G_1(\psi_\ell)^*\in\mathcal{L}_2(H;\mathcal{H})$ denotes the adjoint operator of $G_1(\psi_\ell)\in\mathcal{L}_2(\mathcal{H};H)$, i.e., $\langle G_1(\psi_\ell)^*\varphi,h\rangle_{\mathcal{H}}=\langle\varphi,G_1(\psi_\ell)h\rangle_H$ for all $\varphi\in H$, $h\in\mathcal{H}$ and $\ell\in\{1,2\}$,

$$\begin{aligned}&= \sum_{j\in\mathcal{I}}\Big\langle G_1(\psi_1)^*G_1(\psi_1)f_j,\sum_{k\in\mathcal{I}}\langle G_1(\psi_2)^*G_1(\psi_2)f_j,f_k\rangle_{\mathcal{H}}f_k\Big\rangle_{\mathcal{H}}\\
&= \sum_{j\in\mathcal{I}}\langle G_1(\psi_1)^*G_1(\psi_1)f_j, G_1(\psi_2)^*G_1(\psi_2)f_j\rangle_{\mathcal{H}},\end{aligned}$$

since $(f_k)_{k\in\mathcal{I}}$ is an orthonormal basis of $\mathcal{H}=Q^{1/2}U$. Now we use the Cauchy–Schwarz inequality and obtain

$$\begin{aligned}&\le \sum_{j\in\mathcal{I}}\|G_1(\psi_1)^*G_1(\psi_1)f_j\|_{\mathcal{H}}\|G_1(\psi_2)^*G_1(\psi_2)f_j\|_{\mathcal{H}}\\
&\le \|G_1(\psi_1)^*G_1(\psi_1)\|_{\mathcal{L}_2(\mathcal{H};\mathcal{H})}\|G_1(\psi_2)^*G_1(\psi_2)\|_{\mathcal{L}_2(\mathcal{H};\mathcal{H})}\\
&= \|G_1(\psi_1)\|^2_{\mathcal{L}_4(\mathcal{H};H)}\|G_1(\psi_2)\|^2_{\mathcal{L}_4(\mathcal{H};H)}\\
&\le \|G_1\|^4_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}\|\psi_1\|^2_V\|\psi_2\|^2_V.\end{aligned}$$

In this calculation the equality within the last three lines can be justified as follows: For $\vartheta\in V$, we denote by $(s_n(G_1(\vartheta)))_{n\in\mathbb{N}}$ the singular values of the operator $G_1(\vartheta)$, i.e., the eigenvalues of the operator $(G_1(\vartheta)^* G_1(\vartheta))^{1/2}$. Then the singular values $(s_n(G_1(\vartheta)^* G_1(\vartheta)))_{n\in\mathbb{N}}$ of the operator $G_1(\vartheta)^* G_1(\vartheta)$ are given by

$$s_n(G_1(\vartheta)^* G_1(\vartheta)) = s_n(G_1(\vartheta))^2, \quad n \in \mathbb{N}.$$

Hence, we obtain the following equality

$$\begin{aligned}\|G_1(\vartheta)^* G_1(\vartheta)\|^2_{\mathcal{L}_2(\mathcal{H};\mathcal{H})} &= \sum_{n\in\mathbb{N}} s_n(G_1(\vartheta)^* G_1(\vartheta))^2 \\ &= \sum_{n\in\mathbb{N}} s_n(G_1(\vartheta))^4 = \|G_1(\vartheta)\|^4_{\mathcal{L}_4(\mathcal{H};H)}.\end{aligned}$$

We have shown that

$$\|(G_1 \otimes G_1)(\psi_1 \otimes \psi_2)q\|_{H^{(2)}} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} \|\psi_1 \otimes \psi_2\|_{V^{(2)}}$$

for all $\psi_1, \psi_2 \in V$. Since the set $\operatorname{span}\{\psi_1 \otimes \psi_2 : \psi_1, \psi_2 \in V\}$ is dense in $V^{(2)}$ the first claim follows: $(G_1 \otimes G_1)(\cdot)q \in \mathcal{L}\left(V^{(2)}; H^{(2)}\right)$ with

$$\|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(V^{(2)};H^{(2)})} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}.$$

The second and the third assertion can be proven similarly: For $\psi \in V$

$$\begin{aligned}\|(G_1(\psi) \otimes G_2)q\|^2_{H^{(2)}} &= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \langle G_1(\psi)f_j \otimes G_2 f_j, G_1(\psi)f_k \otimes G_2 f_k\rangle_{H^{(2)}} \\ &= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \langle G_1(\psi)f_j, G_1(\psi)f_k\rangle_H \langle G_2 f_j, G_2 f_k\rangle_H = \sum_{j\in\mathcal{I}} \langle G_1(\psi)^* G_1(\psi) f_j, G_2^* G_2 f_j\rangle_{\mathcal{H}} \\ &\le \|G_1(\psi)^* G_1(\psi)\|_{\mathcal{L}_2(\mathcal{H};\mathcal{H})} \|G_2^* G_2\|_{\mathcal{L}_2(\mathcal{H};\mathcal{H})} = \|G_1(\psi)\|^2_{\mathcal{L}_4(\mathcal{H};H)} \|G_2\|^2_{\mathcal{L}_4(\mathcal{H};H)} \\ &\le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} \|G_2\|^2_{\mathcal{L}_4(\mathcal{H};H)} \|\psi\|^2_V.\end{aligned}$$

By symmetry of Q, it holds $\|(G_2 \otimes G_1(\psi))q\|_{H^{(2)}} = \|(G_1(\psi) \otimes G_2)q\|_{H^{(2)}}$ for all $\psi \in V$. This shows (ii) and for (iii) we obtain

$$\begin{aligned}\|(G_2 \otimes G_2)q\|^2_{H^{(2)}} &= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \langle G_2 f_j \otimes G_2 f_j, G_2 f_k \otimes G_2 f_k\rangle_{H^{(2)}} \\ &= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \langle G_2 f_j, G_2 f_k\rangle^2_H = \sum_{j\in\mathcal{I}} \|G_2^* G_2 f_j\|^2_{\mathcal{H}} = \|G_2^* G_2\|^2_{\mathcal{L}_2(\mathcal{H};\mathcal{H})} = \|G_2\|^4_{\mathcal{L}_4(\mathcal{H};H)}. \ \square\end{aligned}$$

Lemma 2.7 relates the concepts of weak and mild solutions of stochastic partial differential equations, cf. [9, Section 9.3], and will provide the basis for establishing the connection between the second moment of the mild solution and a space-time variational problem. In order to state it, we have to define the differential operator ∂_t first. For a vector-valued function $u\colon \mathbb{T} \to H$ taking values in a Hilbert space H we define the distributional derivative $\partial_t u$ as the H-valued distribution satisfying

$$\langle (\partial_t u)(w), \varphi\rangle_H = -\int_0^T \frac{\mathrm{d}w}{\mathrm{d}t}(t) \langle u(t), \varphi\rangle_H \quad \forall \varphi \in H,$$

for all $w \in C_0^\infty(\mathbb{T};\mathbb{R})$, cf. [4, Definition 3 in §XVIII.1].

Lemma 2.7. *Let the conditions (1)–(2) of Assumption 2.1 be satisfied and let X be the mild solution to* (2.2). *Then it holds $\mathbb{P}$-almost surely that*

$$\langle X, (-\partial_t + A^*)v\rangle_{L^2(\mathbb{T};H)} = \langle X_0, v(0)\rangle_H + \int_0^T \langle v(t), G(X(t))\,\mathrm{d}L(t)\rangle_H$$

for all $v \in C^1_{0,\{T\}}(\mathbb{T};\mathcal{D}(A^*)) := \{w \in C^1(\mathbb{T},\mathcal{D}(A^*)) : w(T) = 0\}$.

Proof. This equality follows from the equivalence of mild and weak solutions, see [9, Theorem 9.15 and (9.20)]. □

3. The second moment

After having introduced the stochastic partial differential equation of interest and its mild solution in Section 2, the aim of this section is to derive a well-posed deterministic variational problem which is satisfied by the second moment of the mild solution.

The second moment of an H-valued random variable $X \in L^2(\Omega; H)$ is denoted by $\mathbb{M}^{(2)}X := \mathbb{E}[X \otimes X] \in H^{(2)}$. It follows immediately from the definition of the mild solution that its second moment is an element of the tensor space $L^2(\mathbb{T}; H) \otimes L^2(\mathbb{T}; H)$. Under the assumptions made above we can prove even more regularity. We introduce the Bochner space $\mathcal{X} := L^2(\mathbb{T}; V)$ and the tensor space $\mathcal{X}^{(2)} := \mathcal{X} \otimes \mathcal{X}$.

Theorem 3.1. *Let all conditions of Assumption 2.1 be satisfied. Then the second moment $\mathbb{M}^{(2)}X$ of the mild solution X defined in* (2.3) *satisfies $\mathbb{M}^{(2)}X \in \mathcal{X}^{(2)}$.*

Proof. First, we remark that

$$\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(2)}} = \|\mathbb{E}[X \otimes X]\|_{\mathcal{X}^{(2)}} \le \mathbb{E}\|X \otimes X\|_{\mathcal{X}^{(2)}} = \mathbb{E}\left[\|X\|_{\mathcal{X}}^2\right].$$

Hence, we may estimate as follows:

$$\begin{aligned}
&\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(2)}} \le \mathbb{E}\int_0^T \left\| S(t)X_0 + \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|_V^2 \mathrm{d}t \\
&\le 2\,\mathbb{E}\int_0^T \left[\|S(t)X_0\|_V^2 + \left\|\int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|_V^2\right] \mathrm{d}t \\
&= 2\,\mathbb{E}\left[\int_0^T \|A^{\frac12}S(t)X_0\|_H^2\,\mathrm{d}t\right] \quad + 2\int_0^T \mathbb{E}\left\|\int_0^t A^{\frac12}S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|_H^2 \mathrm{d}t.
\end{aligned}$$

Since the generator $-A$ of the semigroup $(S(t), t \ge 0)$ is self-adjoint and negative definite, we can bound the first integral from above by using the inequality

$$\int_0^T \|A^{\frac12}S(t)\varphi\|_H^2\,\mathrm{d}t \le \frac{1}{2}\|\varphi\|_H^2, \qquad \varphi \in H, \tag{3.1}$$

and for the second term we use Itô's isometry, cf. [9, Corollary 8.17], as well as the affine structure of the operator G to obtain

$$\begin{aligned}
\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(2)}} &\le \mathbb{E}\|X_0\|_H^2 + 2\int_0^T \mathbb{E}\int_0^t \|A^{\frac12}S(t-s)G(X(s))\|_{\mathcal{L}_2(\mathcal{H};H)}^2\,\mathrm{d}s\,\mathrm{d}t \\
&\le \mathbb{E}\|X_0\|_H^2 + 4\int_0^T\int_0^t \|A^{\frac12}S(t-s)G_2\|_{\mathcal{L}_2(\mathcal{H};H)}^2\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + 4\int_0^T \mathbb{E}\int_0^t \|A^{\frac12}S(t-s)G_1(X(s))\|_{\mathcal{L}_2(\mathcal{H};H)}^2\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

By Assumption 2.1 (1)–(3) as well as Theorem 2.3 there exists a regularity exponent $r \in [0,1]$ such that the mild solution satisfies $X \in L^\infty(\mathbb{T}; L^2(\Omega; \dot{H}^r))$. In addition, by part (4) of Assumption 2.1 it holds $A^{1/2}S(\cdot)G_1 \in L^2(\mathbb{T}; \mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H)))$. Then we estimate as follows,

$$\begin{aligned}\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(2)}} \le \mathbb{E}\|X_0\|_H^2 &+ 4\sum_{n\in\mathbb{N}}\int_0^T\int_0^t \|A^{\frac12}S(t-s)G_2 f_n\|_H^2\,\mathrm{d}s\,\mathrm{d}t \\ &+ 4\int_0^T\int_0^t \|A^{\frac12}S(t-s)G_1\|^2_{\mathcal{L}(\dot H^r;\mathcal{L}_2(\mathcal{H};H))}\mathbb{E}\|X(s)\|^2_{\dot H^r}\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

for an orthonormal basis $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$. Applying (3.1) again yields

$$\begin{aligned}\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(2)}} \le \|X_0\|^2_{L^2(\Omega;H)} &+ 2T\|G_2\|^2_{\mathcal{L}_2(\mathcal{H};H)} \\ &+ 4T\|X\|^2_{L^\infty(\mathbb{T};L^2(\Omega;\dot H^r))}\|A^{\frac12}S(\cdot)G_1\|^2_{L^2(\mathbb{T};\mathcal{L}(\dot H^r;\mathcal{L}_2(\mathcal{H};H)))},\end{aligned}$$

which is finite under our assumptions and completes the proof. $\square$

We introduce the spaces $H^1_{0,\{T\}}(\mathbb{T};V^*) := \{v\in H^1(\mathbb{T};V^*) : v(T)=0\}$ as well as $\mathcal{Y} := L^2(\mathbb{T};V)\cap H^1_{0,\{T\}}(\mathbb{T};V^*)$. $\mathcal{Y}$ is a Hilbert space with respect to the inner product

$$\langle v_1,v_2\rangle_{\mathcal{Y}} := \langle v_1,v_2\rangle_{L^2(\mathbb{T};V)} + \langle \partial_t v_1,\partial_t v_2\rangle_{L^2(\mathbb{T};V^*)},\quad v_1,v_2\in\mathcal{Y}.$$

Moreover, we obtain the following continuous embedding.

Lemma 3.2. *It holds that $\mathcal{Y}\hookrightarrow C^0(\mathbb{T};H)$ with embedding constant $C\le 1$, i.e., $\sup_{s\in\mathbb{T}}\|v(s)\|_H \le \|v\|_{\mathcal{Y}}$ for every $v\in\mathcal{Y}$.*

Proof. For every $v\in\mathcal{Y} = L^2(\mathbb{T};V)\cap H^1_{0,\{T\}}(\mathbb{T};V^*)$ we have the relation

$$\|v(r)\|_H^2 - \|v(s)\|_H^2 = \int_s^r 2\ {}_{V^*}\langle\partial_t v(t), v(t)\rangle_V\,\mathrm{d}t,\quad r,s\in\mathbb{T},\ r>s,$$

cf. [4, §XVIII.1, Theorem 2]. Choosing $r=T$ and observing that $v(T)=0$ leads to

$$\|v(s)\|_H^2 \le 2\,\|\partial_t v\|_{L^2(\mathbb{T};V^*)}\|v\|_{L^2(\mathbb{T};V)} \le \|\partial_t v\|^2_{L^2(\mathbb{T};V^*)} + \|v\|^2_{L^2(\mathbb{T};V)} = \|v\|^2_{\mathcal{Y}}. \quad\square$$

The dual spaces of $\mathcal{X}$ and $\mathcal{Y}$ with respect to the pivot space $L^2(\mathbb{T};H)$ are denoted by $\mathcal{X}^*$ and $\mathcal{Y}^*$, respectively. For the tensor spaces $\mathcal{X}^{(2)}$ and $\mathcal{Y}^{(2)} := \mathcal{Y}\otimes\mathcal{Y}$ the dual spaces $\mathcal{X}^{(2)*}$ and $\mathcal{Y}^{(2)*}$ are taken with respect to the pivot space $L^2(\mathbb{T};H)^{(2)} := L^2(\mathbb{T};H)\otimes L^2(\mathbb{T};H)$.

In the deterministic equation satisfied by the second moment the *diagonal trace operator* δ will play an important role. For $w\in C^0(\mathbb{T}\times\mathbb{T};\mathbb{R})$ we define the diagonal trace operator $\delta\colon C^0(\mathbb{T}\times\mathbb{T};\mathbb{R})\to C^0(\mathbb{T};\mathbb{R})\subset L^1(\mathbb{T};\mathbb{R})$ by

$$\delta(w)(t) := w(t,t)\quad \forall t\in\mathbb{T}.$$

This operator admits a unique continuous linear extension to an operator acting on $L^2(\mathbb{T};\mathbb{R})^{(2)}$, $\delta\colon L^2(\mathbb{T};\mathbb{R})^{(2)}\to L^1(\mathbb{T};\mathbb{R})$. For a function

$$u\in\mathcal{U} := \{u\in C^0(\mathbb{T}\times\mathbb{T};\mathbb{R}) \,|\, \exists u_1,u_2\in C^0(\mathbb{T};\mathbb{R}) : u(s,t) := u_1(s)u_2(t)\quad \forall s,t\in\mathbb{T}\}$$

we calculate

$$\|\delta(u)\|_{L^1(\mathbb{T};\mathbb{R})} = \int_0^T |u_1(t)u_2(t)|\,\mathrm{d}t \le \|u_1\|_{L^2(\mathbb{T};\mathbb{R})}\|u_2\|_{L^2(\mathbb{T};\mathbb{R})} = \|u\|_{L^2(\mathbb{T}\times\mathbb{T};\mathbb{R})}.$$

The set of linear combinations $\operatorname{span}(\mathcal{U})$ is a dense subset of $L^2(\mathbb{T}\times\mathbb{T};\mathbb{R})$ and, moreover, the spaces $L^2(\mathbb{T}\times\mathbb{T};\mathbb{R})$ and $L^2(\mathbb{T};\mathbb{R})^{(2)}$ are isometrically isomorphic. Hence,

$$\delta \in \mathcal{L}\big(L^2(\mathbb{T};\mathbb{R})^{(2)}; L^1(\mathbb{T};\mathbb{R})\big), \qquad \|\delta\|_{\mathcal{L}(L^2(\mathbb{T};\mathbb{R})^{(2)};L^1(\mathbb{T};\mathbb{R}))} \le 1.$$

We next extend the definition of the diagonal trace operator to vector-valued functions. For a separable Hilbert space H we consider the tensor product between the Banach space $L^1(\mathbb{T};\mathbb{R})$ and $H^{(2)}$ with respect to the projective norm, cf. [11, Section 2.1]. We define $\delta\colon L^2(\mathbb{T};\mathbb{R})^{(2)}\otimes H^{(2)} \to L^1(\mathbb{T};\mathbb{R})\otimes H^{(2)}$ for generating functions $u = v\otimes\varphi$, $v\in L^2(\mathbb{T};\mathbb{R})^{(2)}$, $\varphi\in H^{(2)}$, by

$$\delta(u) := \delta(v)\otimes\varphi \in L^1(\mathbb{T};\mathbb{R})\otimes H^{(2)}$$

and extend continuously. In this way we obtain a bounded linear operator

$$\delta \in \mathcal{L}\big(L^2(\mathbb{T};H)^{(2)}; L^1\big(\mathbb{T};H^{(2)}\big)\big), \qquad \|\delta\|_{\mathcal{L}(L^2(\mathbb{T};H)^{(2)};L^1(\mathbb{T};H^{(2)}))} \le 1, \tag{3.2}$$

since $L^2(\mathbb{T};H)^{(2)} \cong L^2(\mathbb{T};\mathbb{R})^{(2)}\otimes H^{(2)}$ and $L^1\left(\mathbb{T};H^{(2)}\right) \cong L^1(\mathbb{T};\mathbb{R})\otimes H^{(2)}$ – for the latter identification see [11, Section 2.3]. Note that the restriction $\delta_{\mathcal{Y}}$ of the diagonal trace operator δ to the subspace $\mathcal{Y}^{(2)} \subset L^2(\mathbb{T};H)^{(2)}$ is mapping to $C^0\left(\mathbb{T};H^{(2)}\right)$, since for $v_1, v_2 \in \mathcal{Y}$ we obtain

$$\begin{aligned}\|\delta_{\mathcal{Y}}(v_1\otimes v_2)\|_{C^0(\mathbb{T};H^{(2)})} &= \sup_{t\in\mathbb{T}}\|v_1(t)\otimes v_2(t)\|_{H^{(2)}} \le \sup_{s\in\mathbb{T}}\|v_1(s)\|_H \sup_{t\in\mathbb{T}}\|v_2(t)\|_H \\ &\le \|v_1\|_{\mathcal{Y}}\|v_2\|_{\mathcal{Y}} = \|v_1\otimes v_2\|_{\mathcal{Y}^{(2)}}\end{aligned}$$

by Lemma 3.2 above. Therefore,

$$\delta_{\mathcal{Y}} \in \mathcal{L}\big(\mathcal{Y}^{(2)}; C^0\big(\mathbb{T};H^{(2)}\big)\big), \qquad \|\delta_{\mathcal{Y}}\|_{\mathcal{L}(\mathcal{Y}^{(2)};C^0(\mathbb{T};H^{(2)}))} \le 1, \tag{3.3}$$

and we may define its adjoint operator $\delta_{\mathcal{Y}}^* \in \mathcal{L}\big(C^0\big(\mathbb{T};H^{(2)}\big)^*; \mathcal{Y}^{(2)*}\big)$. Then the vector $\delta_{\mathcal{Y}}^*(\delta(u))$ is a well-defined element in the dual space $\mathcal{Y}^{(2)*}$ for all $u\in L^2(\mathbb{T};H)^{(2)}$, since $L^1\left(\mathbb{T};H^{(2)}\right) \subset C^0\left(\mathbb{T};H^{(2)}\right)^*$.

The following proposition establishes the finiteness of all terms in the deterministic variational problem that we shall derive in Theorem 3.5 below.

Proposition 3.3. *For the kernel $q\in U^{(2)}$ defined in* (2.6) *and an affine operator $G(\cdot) = G_1(\cdot) + G_2$ satisfying condition (2) of Assumption 2.1 the following operators are in the dual space of $\mathcal{Y}^{(2)}$:*

$$\begin{aligned}\delta_{\mathcal{Y}}^*(\delta((G_1\otimes G_1)(u)q)) &\in \mathcal{Y}^{(2)*} \quad \forall u\in\mathcal{X}^{(2)},\\ \delta_{\mathcal{Y}}^*((G_1(u)\otimes G_2)q) &\in \mathcal{Y}^{(2)*} \quad \forall u\in\mathcal{X},\\ \delta_{\mathcal{Y}}^*((G_2\otimes G_1(u))q) &\in \mathcal{Y}^{(2)*} \quad \forall u\in\mathcal{X},\\ \delta_{\mathcal{Y}}^*((G_2\otimes G_2)q) &\in \mathcal{Y}^{(2)*}.\end{aligned}$$

Moreover, the linear operator $\delta_{\mathcal{Y}}^*(\delta((G_1\otimes G_1)(\cdot)q))\colon \mathcal{X}^{(2)}\to\mathcal{Y}^{(2)*}$ is bounded with

$$\|\delta_{\mathcal{Y}}^*(\delta((G_1\otimes G_1)(\cdot)q))\|_{\mathcal{L}(\mathcal{X}^{(2)};\mathcal{Y}^{(2)*})} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}.$$

Proof. As mentioned above, it is enough to show that $(G_1\otimes G_1)(u)q \in L^2(\mathbb{T};H)^{(2)}$ for all $u\in\mathcal{X}^{(2)}$ in order to prove the first claim. By Lemma 2.6 (i), we have that $(G_1\otimes G_1)(\cdot)q \in \mathcal{L}\left(V^{(2)};H^{(2)}\right)$, which justifies that

$$(G_1\otimes G_1)(\cdot)q \in \mathcal{L}\big(\mathcal{X}^{(2)}; L^2(\mathbb{T};H)^{(2)}\big)$$

with

$$\|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(\mathcal{X}^{(2)};L^2(\mathbb{T};H)^{(2)})} = \|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(V^{(2)};H^{(2)})}, \tag{3.4}$$

since $\mathcal{X}^{(2)} \cong L^2(\mathbb{T};\mathbb{R})^{(2)} \otimes V^{(2)}$. Furthermore, we obtain for $u \in \mathcal{X}^{(2)}$

$$\begin{aligned}
&\|\delta^*_{\mathcal{Y}}(\delta((G_1 \otimes G_1)(u)q))\|_{\mathcal{Y}^{(2)*}} \\
&\qquad \le \|\delta^*_{\mathcal{Y}}\|_{\mathcal{L}(C^0(\mathbb{T};H^{(2)})^*;\mathcal{Y}^{(2)*})} \|\delta((G_1 \otimes G_1)(u)q)\|_{C^0(\mathbb{T};H^{(2)})^*} \\
&\qquad \le \|\delta((G_1 \otimes G_1)(u)q)\|_{L^1(\mathbb{T};H^{(2)})},
\end{aligned}$$

since $\delta^*_{\mathcal{Y}}$ inherits the bound of the operator norm in (3.3) from $\delta_{\mathcal{Y}}$ and, in addition, $\|w\|_{C^0(\mathbb{T};H^{(2)})^*} \le \|w\|_{L^1(\mathbb{T};H^{(2)})}$ for all $w \in L^1\big(\mathbb{T};H^{(2)}\big)$. By applying (3.2) and (3.4) we obtain

$$\begin{aligned}
\|\delta^*_{\mathcal{Y}}(\delta((G_1 \otimes G_1)(u)q))\|_{\mathcal{Y}^{(2)*}} &\le \|\delta\|_{\mathcal{L}(L^2(\mathbb{T};H)^{(2)};L^1(\mathbb{T};H^{(2)}))} \|(G_1 \otimes G_1)(u)q\|_{L^2(\mathbb{T};H)^{(2)}} \\
&\le \|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(\mathcal{X}^{(2)};L^2(\mathbb{T};H)^{(2)})} \|u\|_{\mathcal{X}^{(2)}} \\
&= \|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(V^{(2)};H^{(2)})} \|u\|_{\mathcal{X}^{(2)}} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} \|u\|_{\mathcal{X}^{(2)}}.
\end{aligned}$$

Here, we used the bound on the operator norm in Lemma 2.6 (i) for the last estimate.

To see that $\delta^*_{\mathcal{Y}}((G_1(u) \otimes G_2)q)$ and $\delta^*_{\mathcal{Y}}((G_2 \otimes G_1(u))q)$ are in the dual space $\mathcal{Y}^{(2)*}$ for every $u \in \mathcal{X}$, we may proceed in the same way using part (ii) of Lemma 2.6 yielding

$$(G_1(\cdot) \otimes G_2)q \in \mathcal{L}\big(\mathcal{X}; L^2\big(\mathbb{T};H^{(2)}\big)\big), \quad (G_2 \otimes G_1(\cdot))q \in \mathcal{L}\big(\mathcal{X}; L^2\big(\mathbb{T};H^{(2)}\big)\big).$$

Hence, $(G_1(u) \otimes G_2)q, (G_2 \otimes G_1(u))q \in L^2\big(\mathbb{T};H^{(2)}\big)$ for every $u \in \mathcal{X}$ and the second as well as the third claim follow since $L^2\big(\mathbb{T};H^{(2)}\big) \subset L^1\big(\mathbb{T};H^{(2)}\big)$.

Finally, by Lemma 2.6 (iii), $(G_2 \otimes G_2)q \in H^{(2)}$ and, therefore, it is a constant function in $L^1\left(\mathbb{T};H^{(2)}\right)$ and the last assertion follows. $\square$

Remark 3.4. In the additive case one may relax the assumptions on the operators A and Q made in [7, Lemma 4.1]. In fact, the condition $\operatorname{tr}(AQ) < +\infty$ is not necessary for the term denoted by $\delta \,\tilde{\otimes}\, q$ in [7] to be in the dual space $\mathcal{Y}^{(2)*}$, since for $v_1, v_2 \in L^2(\mathbb{T};H) \supset \mathcal{Y}$ we may estimate as follows

$$\begin{aligned}
|\langle \delta \,\tilde{\otimes}\, q, v_1 \otimes v_2\rangle_{L^2(\mathbb{T};H)^{(2)}}| &= \Big| \int_0^T \langle q, v_1(t) \otimes v_2(t)\rangle_{H^{(2)}} \,\mathrm{d}t \Big| \\
&\le \|q\|_{H^{(2)}} \int_0^T \|v_1(t)\|_H \|v_2(t)\|_H \,\mathrm{d}t \le \|q\|_{H^{(2)}} \|v_1\|_{L^2(\mathbb{T};H)} \|v_2\|_{L^2(\mathbb{T};H)}.
\end{aligned}$$

The calculation above shows that $\delta \,\tilde{\otimes}\, q$ is even an element of $L^2(\mathbb{T};H)^{(2)*} \subset \mathcal{Y}^{(2)*}$ without the assumption $\operatorname{tr}(AQ) < +\infty$.

Finally, we define the bilinear forms $\mathcal{B} \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$,

$$\mathcal{B}(u,v) := \int_0^T {}_V\langle u(t), (-\partial_t + A^*)v(t)\rangle_{V^*} \,\mathrm{d}t = {}_{\mathcal{X}}\langle u, (-\partial_t + A^*)v\rangle_{\mathcal{X}^*}, \tag{3.5}$$

as well as $\mathcal{B}^{(2)} \colon \mathcal{X}^{(2)} \times \mathcal{Y}^{(2)} \to \mathbb{R}$,

$$\begin{aligned}
\mathcal{B}^{(2)}(u,v) &:= \int_0^T \int_0^T {}_{V^{(2)}}\langle u(t_1,t_2), ((-\partial_t + A^*) \otimes (-\partial_t + A^*))\, v(t_1,t_2)\rangle_{V^{(2)*}} \,\mathrm{d}t_1 \,\mathrm{d}t_2 \\
&= {}_{\mathcal{X}^{(2)}}\langle u, (-\partial_t + A^*)^{(2)} v\rangle_{\mathcal{X}^{(2)*}}, \qquad (3.6)
\end{aligned}$$

and we introduce the mean function m of the mild solution X in (2.3), i.e.,

$$m(t) := \mathbb{E}X(t) = S(t)\mathbb{E}X_0, \quad t \in \mathbb{T}. \tag{3.7}$$

Note that due to the martingale property of the stochastic integral the mean function depends only on the initial value X_0 and not on the operator G. Furthermore, applying inequality (3.1) shows the regularity $m \in \mathcal{X}$, and m can be interpreted as the unique function satisfying

$$m \in \mathcal{X} : \quad \mathcal{B}(m, v) = \langle \mathbb{E}X_0, v(0)\rangle_H \quad \forall v \in \mathcal{Y}. \tag{3.8}$$

Well-posedness of the problem (3.8) above follows from [13, Theorem 2.3].

With these definitions and preliminaries we are now able to show that the second moment of the mild solution solves a deterministic variational problem.

Theorem 3.5. *Let all conditions of Assumption 2.1 be satisfied and let X be the mild solution to* (2.2). *Then the second moment $\mathbb{M}^{(2)}X \in \mathcal{X}^{(2)}$ solves the following variational problem*

$$u \in \mathcal{X}^{(2)} : \quad \widetilde{\mathcal{B}}^{(2)}(u, v) = f(v) \quad \forall v \in \mathcal{Y}^{(2)}, \tag{3.9}$$

where for $u \in \mathcal{X}^{(2)}$ and $v \in \mathcal{Y}^{(2)}$

$$\begin{aligned} \widetilde{\mathcal{B}}^{(2)}(u, v) &:= \mathcal{B}^{(2)}(u, v) - {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*(\delta((G_1 \otimes G_1)(u)q)), v\rangle_{\mathcal{Y}^{(2)}}, \\ f(v) &:= \langle \mathbb{M}^{(2)}X_0, v(0,0)\rangle_{H^{(2)}} + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*((G_1(m) \otimes G_2)q), v\rangle_{\mathcal{Y}^{(2)}} \\ &\quad + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*((G_2 \otimes G_1(m) + G_2 \otimes G_2)q), v\rangle_{\mathcal{Y}^{(2)}} \end{aligned} \tag{3.10}$$

with the mean function $m \in \mathcal{X}$ defined in (3.7).

Proof. Let $v_1, v_2 \in C^1_{0,\{T\}}(\mathbb{T}; \mathcal{D}(A^*)) = \{\phi \in C^1(\mathbb{T}; \mathcal{D}(A^*)) : \phi(T) = 0\}$. Then, by (3.6), we obtain

$$\begin{aligned} \mathcal{B}^{(2)}\big(\mathbb{M}^{(2)}X, v_1 \otimes v_2\big) &= {}_{\mathcal{X}^{(2)}}\langle \mathbb{M}^{(2)}X, (-\partial_t + A^*)^{(2)}(v_1 \otimes v_2)\rangle_{\mathcal{X}^{(2)*}} \\ &= \mathbb{E}\, {}_{\mathcal{X}^{(2)}}\langle X \otimes X, (-\partial_t + A^*)^{(2)}(v_1 \otimes v_2)\rangle_{\mathcal{X}^{(2)*}} \\ &= \mathbb{E}\left[\langle X, (-\partial_t + A^*)v_1\rangle_{L^2(\mathbb{T};H)} \langle X, (-\partial_t + A^*)v_2\rangle_{L^2(\mathbb{T};H)}\right]. \end{aligned}$$

Because of the regularity of v_1 and v_2 we may take the inner product on $L^2(\mathbb{T}; H)$ instead of the dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$. Now, since X is the mild solution of (2.2), Lemma 2.7 yields

$$\begin{aligned} \mathcal{B}^{(2)}\big(\mathbb{M}^{(2)}X, v_1 \otimes v_2\big) &= \mathbb{E}\Big[\Big(\langle X_0, v_1(0)\rangle_H + \int_0^T \langle v_1(s), G(X(s))\, \mathrm{d}L(s)\rangle_H\Big) \\ &\qquad \cdot \Big(\langle X_0, v_2(0)\rangle_H + \int_0^T \langle v_2(t), G(X(t))\, \mathrm{d}L(t)\rangle_H\Big)\Big] \\ &= \mathbb{E}\left[\langle X_0, v_1(0)\rangle_H \langle X_0, v_2(0)\rangle_H\right] \\ &\quad + \mathbb{E}\Big[\langle X_0, v_1(0)\rangle_H \int_0^T \langle v_2(t), G(X(t))\, \mathrm{d}L(t)\rangle_H\Big] \\ &\quad + \mathbb{E}\Big[\langle X_0, v_2(0)\rangle_H \int_0^T \langle v_1(s), G(X(s))\, \mathrm{d}L(s)\rangle_H\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \langle v_2(t), G(X(t))\, \mathrm{d}L(t)\rangle_H \int_0^T \langle v_1(s), G(X(s))\, \mathrm{d}L(s)\rangle_H\Big]. \end{aligned}$$

The $\mathcal{F}_0$-measurability of $X_0 \in L^2(\Omega; H)$, along with the martingale property of the stochastic integral, imply that the second and the third term vanish: For $\ell \in \{1,2\}$ we define the $\mathcal{L}_2(\mathcal{H};\mathbb{R})$-valued stochastic process Ψ_ℓ by

$$\Psi_\ell(t)\colon w \mapsto \langle v_\ell(t), G(X(t))w\rangle_H \quad \forall w \in \mathcal{H}$$

for $t \in \mathbb{T}$, $\mathbb{P}$-almost surely. Then we obtain $\|\Psi_\ell(t)\|^2_{\mathcal{L}_2(\mathcal{H};\mathbb{R})} = \|G(X(t))^* v_\ell(t)\|^2_{\mathcal{H}}$ $\mathbb{P}$-almost surely with the adjoint $G(X(t))^* \in \mathcal{L}_2(H;\mathcal{H})$ of $G(X(t))$ and

$$\begin{aligned}\mathbb{E}\Big[\langle X_0, v_\ell(0)\rangle_H \int_0^T \langle v_\ell(t), G(X(t))\,\mathrm{d}L(t)\rangle_H\Big] &= \mathbb{E}\Big[\langle X_0, v_\ell(0)\rangle_H \int_0^T \Psi_\ell(t)\,\mathrm{d}L(t)\Big] \\ &= \mathbb{E}\bigg[\langle X_0, v_\ell(0)\rangle_H \mathbb{E}\Big[\int_0^T \Psi_\ell(t)\,\mathrm{d}L(t)\,\Big|\,\mathcal{F}_0\Big]\bigg] = 0\end{aligned}$$

by the definition of the weak stochastic integral, cf. [9, p. 151], and the martingale property of the stochastic integral. For the first term we calculate

$$\begin{aligned}\mathbb{E}\left[\langle X_0, v_1(0)\rangle_H \langle X_0, v_2(0)\rangle_H\right] &= \mathbb{E}\left[\langle X_0 \otimes X_0, v_1(0)\otimes v_2(0)\rangle_{H^{(2)}}\right] \\ &= \langle \mathbb{M}^{(2)} X_0, (v_1\otimes v_2)(0,0)\rangle_{H^{(2)}}.\end{aligned}$$

Finally, the predictability of X together with the continuity assumptions on G imply the predictability of $G(X)$ and we may use Lemma 2.5 for the last term yielding

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T \langle v_2(t), G(X(t))\,\mathrm{d}L(t)\rangle_H \int_0^T \langle v_1(s), G(X(s))\,\mathrm{d}L(s)\rangle_H\Big] \\ &\quad= \int_0^T \langle v_1(t)\otimes v_2(t), \mathbb{E}\left[G(X(t))\otimes G(X(t))\right] q\rangle_{H^{(2)}}\,\mathrm{d}t \\ &\quad= \int_0^T \langle (v_1\otimes v_2)(t,t), \mathbb{E}\left[(G_1\otimes G_1)(X(t)\otimes X(t))\right] q\rangle_{H^{(2)}}\,\mathrm{d}t \\ &\qquad+ \int_0^T \langle (v_1\otimes v_2)(t,t), (\mathbb{E}\left[G_1(X(t))\right]\otimes G_2) q\rangle_{H^{(2)}}\,\mathrm{d}t \\ &\qquad+ \int_0^T \langle (v_1\otimes v_2)(t,t), (G_2 \otimes \mathbb{E}\left[G_1(X(t))\right]) q\rangle_{H^{(2)}}\,\mathrm{d}t \\ &\qquad+ \int_0^T \langle (v_1\otimes v_2)(t,t), (G_2\otimes G_2) q\rangle_{H^{(2)}}\,\mathrm{d}t, \\ &\quad= {}_{C^0(\mathbb{T};H^{(2)})^*}\langle \delta((G_1\otimes G_1)(\mathbb{M}^{(2)}X)q), \delta_{\mathcal{Y}}(v_1\otimes v_2)\rangle_{C^0(\mathbb{T};H^{(2)})} \\ &\qquad+ {}_{C^0(\mathbb{T};H^{(2)})^*}\langle (G_1(m)\otimes G_2)q, \delta_{\mathcal{Y}}(v_1\otimes v_2)\rangle_{C^0(\mathbb{T};H^{(2)})} \\ &\qquad+ {}_{C^0(\mathbb{T};H^{(2)})^*}\langle (G_2\otimes G_1(m))q, \delta_{\mathcal{Y}}(v_1\otimes v_2)\rangle_{C^0(\mathbb{T};H^{(2)})} \\ &\qquad+ {}_{C^0(\mathbb{T};H^{(2)})^*}\langle (G_2\otimes G_2)q, \delta_{\mathcal{Y}}(v_1\otimes v_2)\rangle_{C^0(\mathbb{T};H^{(2)})} \\ &\quad= {}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}(\delta((G_1\otimes G_1)(\mathbb{M}^{(2)}X)q)), v_1\otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\ &\qquad+ {}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}((G_1(m)\otimes G_2)q), v_1\otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\ &\qquad+ {}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}((G_2\otimes G_1(m))q), v_1\otimes v_2\rangle_{\mathcal{Y}^{(2)}} + {}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}((G_2\otimes G_2)q), v_1\otimes v_2\rangle_{\mathcal{Y}^{(2)}},\end{aligned}$$

where the last equality holds by Proposition 3.3. Since $C^1_{0,\{T\}}(\mathbb{T};\mathcal{D}(A^*)) \subset \mathcal{Y}$ is a dense subset, the claim follows. $\square$

4. Existence and uniqueness

Before we extend the results of Section 3 for the second moment to the covariance of the mild solution in Section 5, we investigate the well-posedness of the variational problem (3.9) in this section.

For this purpose, we first recall the Nečas theorem, which we quote as it is formulated in [5, Theorem 2.2, p. 422].

Theorem 4.1. *Let H_1 and H_2 be two separable Hilbert spaces and $\mathscr{B}\colon H_1 \times H_2 \to \mathbb{R}$ a continuous bilinear form. Then the variational problem*

$$u \in H_1 : \quad \mathscr{B}(u,v) = {}_{H_2^*}\langle f, v\rangle_{H_2} \quad \forall v \in H_2, \tag{4.1}$$

admits a unique solution $u \in H_1$ for all $f \in H_2^*$, which depends continuously on f, if and only if the bilinear form $\mathscr{B}$ satisfies one of the following equivalent inf-sup conditions:

(1) There exists $\beta > 0$ such that

$$\sup_{v_2 \in H_2 \setminus \{0\}} \frac{\mathscr{B}(v_1, v_2)}{\|v_2\|_{H_2}} \geq \beta \|v_1\|_{H_1} \quad \forall v_1 \in H_1,$$

and for every $0 \neq v_2 \in H_2$ there exists $v_1 \in H_1$ such that $\mathscr{B}(v_1, v_2) \neq 0$.

(2) It holds

$$\inf_{v_1 \in H_1 \setminus \{0\}} \sup_{v_2 \in H_2 \setminus \{0\}} \frac{\mathscr{B}(v_1, v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} > 0, \qquad \inf_{v_2 \in H_2 \setminus \{0\}} \sup_{v_1 \in H_1 \setminus \{0\}} \frac{\mathscr{B}(v_1, v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} > 0.$$

(3) There exists $\beta > 0$ such that

$$\inf_{v_1 \in H_1 \setminus \{0\}} \sup_{v_2 \in H_2 \setminus \{0\}} \frac{\mathscr{B}(v_1, v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} = \inf_{v_2 \in H_2 \setminus \{0\}} \sup_{v_1 \in H_1 \setminus \{0\}} \frac{\mathscr{B}(v_1, v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} = \beta.$$

In addition, the solution u of (4.1) satisfies the stability estimate

$$\|u\|_{H_1} \leq \beta^{-1} \|f\|_{H_2^*}.$$

Therefore, by part (1) of the Nečas theorem above, the deterministic problem of finding $u \in \mathcal{X}^{(2)}$ satisfying (3.9) is well-posed if

$$\inf_{u \in \mathcal{X}^{(2)} \setminus \{0\}} \sup_{v \in \mathcal{Y}^{(2)} \setminus \{0\}} \frac{\widetilde{\mathcal{B}}^{(2)}(u,v)}{\|u\|_{\mathcal{X}^{(2)}} \|v\|_{\mathcal{Y}^{(2)}}} \geq \widetilde{\beta}^{(2)} \tag{4.2}$$

for some constant $\widetilde{\beta}^{(2)} > 0$ and

$$\forall v \in \mathcal{Y}^{(2)} \setminus \{0\} : \quad \sup_{u \in \mathcal{X}^{(2)}} \widetilde{\mathcal{B}}^{(2)}(u,v) > 0. \tag{4.3}$$

First we address the inf-sup condition (4.2) in Section 4.1. The surjectivity (4.3) is proven in Theorem 4.8 in Section 4.2 and well-posedness is deduced.

4.1. **The inf-sup condition.** In order to investigate the inf-sup constant $\widetilde{\beta}^{(2)}$, let us additionally define the following inf-sup constants

$$\beta := \inf_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}, \tag{4.4}$$

$$\beta^* := \inf_{v\in\mathcal{Y}\setminus\{0\}} \sup_{u\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}, \tag{4.5}$$

$$\beta^{(2)} := \inf_{u\in\mathcal{X}^{(2)}\setminus\{0\}} \sup_{v\in\mathcal{Y}^{(2)}\setminus\{0\}} \frac{\mathcal{B}^{(2)}(u,v)}{\|u\|_{\mathcal{X}^{(2)}}\|v\|_{\mathcal{Y}^{(2)}}}, \tag{4.6}$$

$$\beta^{(2)*} := \inf_{v\in\mathcal{Y}^{(2)}\setminus\{0\}} \sup_{u\in\mathcal{X}^{(2)}\setminus\{0\}} \frac{\mathcal{B}^{(2)}(u,v)}{\|u\|_{\mathcal{X}^{(2)}}\|v\|_{\mathcal{Y}^{(2)}}} \tag{4.7}$$

for the bilinear forms $\mathcal{B}$ and $\mathcal{B}^{(2)}$ defined in (3.5) and (3.6), respectively. We immediately obtain the following relation between $\widetilde{\beta}^{(2)}$ and $\beta^{(2)}$.

Lemma 4.2. *For $G_1 \in \mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))$ the inf-sup constant $\widetilde{\beta}^{(2)}$ in* (4.2) *satisfies*

$$\widetilde{\beta}^{(2)} \geq \beta^{(2)} - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}.$$

Proof. To derive the lower bound for $\widetilde{\beta}^{(2)}$, let $u \in \mathcal{X}^{(2)}$. Then

$$\begin{aligned}
\sup_{v\in\mathcal{Y}^{(2)}\setminus\{0\}} \frac{\widetilde{\mathcal{B}}^{(2)}(u,v)}{\|v\|_{\mathcal{Y}^{(2)}}} &= \sup_{v\in\mathcal{Y}^{(2)}\setminus\{0\}} \frac{|\mathcal{B}^{(2)}(u,v) - {}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}(\delta((G_1\otimes G_1)(u)q)), v\rangle_{\mathcal{Y}^{(2)}}|}{\|v\|_{\mathcal{Y}^{(2)}}} \\
&\geq \sup_{v\in\mathcal{Y}^{(2)}\setminus\{0\}} \frac{|\mathcal{B}^{(2)}(u,v)|}{\|v\|_{\mathcal{Y}^{(2)}}} - \sup_{w\in\mathcal{Y}^{(2)}\setminus\{0\}} \frac{|{}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}(\delta((G_1\otimes G_1)(u)q)), w\rangle_{\mathcal{Y}^{(2)}}|}{\|w\|_{\mathcal{Y}^{(2)}}} \\
&\geq \beta^{(2)}\|u\|_{\mathcal{X}^{(2)}} - \|\delta^*_{\mathcal{Y}}(\delta((G_1\otimes G_1)(u)q))\|_{\mathcal{Y}^{(2)*}} \\
&\geq \left(\beta^{(2)} - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}\right)\|u\|_{\mathcal{X}^{(2)}},
\end{aligned}$$

where we used Proposition 3.3 in the last step. □

In order to derive an explicit lower bound for $\widetilde{\beta}^{(2)}$ we investigate $\beta^{(2)}$ first.

Lemma 4.3. *The constants $\beta^{(2)}$ and $\beta^{(2)*}$ in* (4.6) *and* (4.7) *satisfy $\beta^{(2)} \geq \beta^2$ and $\beta^{(2)*} \geq (\beta^*)^2$ with β and β^* defined in* (4.4) *and* (4.5)*, respectively.*

Proof. Applying [1, Lemma 4.4.11] with $X_1 = X_2 = U_1 = U_2 = \mathcal{X}$ and $Y_1 = Y_2 = V_1 = V_2 = \mathcal{Y}$ as well as $\langle u,v\rangle_{X_1\times Y_1} = \langle u,v\rangle_{X_2\times Y_2} = \mathcal{B}(u,v)$ for $u\in\mathcal{X}, v\in\mathcal{Y}$ shows $\beta^{(2)} \geq \beta^2$. Exchanging the roles of $\mathcal{X}$ and $\mathcal{Y}$ yields the second assertion, i.e., that $\beta^{(2)*} \geq (\beta^*)^2$. □

The constant β in (4.4) is known to be positive and, more precisely, we have the following theorem.

Theorem 4.4. *The bilinear form $\mathcal{B}$ in* (3.5) *satisfies the following conditions:*

$$\beta = \inf_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} > 0,$$
$$\forall v\in\mathcal{Y}\setminus\{0\}: \quad \sup_{u\in\mathcal{X}} \mathcal{B}(u,v) > 0.$$

Proof. This result is stated in the second part of [13, Theorem 2.2]. □

Lemma 4.5. *The inf-sup constant* β *in* (4.4) *satisfies* $\beta \geq 1$.

Proof. Combining the results of Theorem 4.4 with the equivalence of (1) and (3) in Theorem 4.1 yields the equality

$$\inf_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} = \inf_{v\in\mathcal{Y}\setminus\{0\}} \sup_{u\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}},$$

i.e., $\beta = \beta^*$. To derive a lower bound for β^*, we proceed as in [12, 14]. Fix $v \in \mathcal{Y} \setminus \{0\}$, and define $u := v - (A^*)^{-1}\partial_t v$, where $(A^*)^{-1}$ is the right-inverse of the surjection A^*. Then $u \in \mathcal{X} = L^2(\mathbb{T}; V)$ since $(A^*)^{-1} \in \mathcal{L}(V^*; V)$ and we calculate as follows:

$$\begin{aligned}
\|u\|_{\mathcal{X}}^2 &= \int_0^T \|u(t)\|_V^2 \,\mathrm{d}t = \int_0^T {}_V\langle u(t), A^* u(t)\rangle_{V^*} \,\mathrm{d}t \\
&= \int_0^T {}_V\langle v(t) - (A^*)^{-1}\partial_t v(t), A^* v(t) - \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t \\
&= \int_0^T {}_V\langle v(t), A^* v(t)\rangle_{V^*} \,\mathrm{d}t + \int_0^T {}_V\langle (A^*)^{-1}\partial_t v(t), \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t \\
&\quad - \int_0^T {}_V\langle v(t), \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t - \int_0^T {}_V\langle (A^*)^{-1}\partial_t v(t), A^* v(t)\rangle_{V^*} \,\mathrm{d}t.
\end{aligned}$$

Now the symmetry of the inner product $\langle \cdot, \cdot \rangle_V$ on V yields

$$\begin{aligned}
{}_V\langle (A^*)^{-1}\partial_t v(t), A^* v(t)\rangle_{V^*} &= \langle (A^*)^{-1}\partial_t v(t), v(t)\rangle_V = \langle v(t), (A^*)^{-1}\partial_t v(t)\rangle_V \\
&= {}_V\langle v(t), \partial_t v(t)\rangle_{V^*},
\end{aligned}$$

and by inserting the identity $A^*(A^*)^{-1}$, using $\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_H^2 = 2\,{}_V\langle v(t), \partial_t v(t)\rangle_{V^*}$ and $v(T) = 0$ we obtain

$$\begin{aligned}
\|u\|_{\mathcal{X}}^2 &= \|v\|_{\mathcal{X}}^2 + \|(A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 - \int_0^T 2\,{}_V\langle v(t), \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t \\
&= \|v\|_{\mathcal{X}}^2 + \|(A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 + \|v(0)\|_H^2 \\
&\geq \|v\|_{\mathcal{X}}^2 + \|(A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 = \|v\|_{\mathcal{X}}^2 + \|\partial_t v\|_{L^2(\mathbb{T};V^*)}^2 = \|v\|_{\mathcal{Y}}^2.
\end{aligned}$$

In the last line we used that $\|w\|_{V^*} = \|(A^*)^{-1}w\|_V$ for every $w \in V^*$ since

$$\begin{aligned}
\|w\|_{V^*} &= \sup_{v\in V\setminus\{0\}} \frac{{}_V\langle v, w\rangle_{V^*}}{\|v\|_V} \\
&= \sup_{v\in V\setminus\{0\}} \frac{{}_V\langle v, A^*((A^*)^{-1}w)\rangle_{V^*}}{\|v\|_V} = \sup_{v\in V\setminus\{0\}} \frac{\langle v, (A^*)^{-1}w\rangle_V}{\|v\|_V} = \|(A^*)^{-1}w\|_V.
\end{aligned}$$

Hence, we obtain for any fixed $v \in \mathcal{Y}$ and $u = v - (A^*)^{-1}\partial_t v$ that $\|u\|_{\mathcal{X}} \geq \|v\|_{\mathcal{Y}}$. In addition, we estimate

$$\begin{aligned}
\mathcal{B}(u,v) &= {}_{\mathcal{X}}\langle u, (-\partial_t + A^*)v\rangle_{\mathcal{X}^*} = {}_{\mathcal{X}}\langle v - (A^*)^{-1}\partial_t v, -\partial_t v + A^* v\rangle_{\mathcal{X}^*} \\
&= \int_0^T {}_V\langle v(t) - (A^*)^{-1}\partial_t v(t), A^*(v(t) - (A^*)^{-1}\partial_t v(t))\rangle_{V^*} \,\mathrm{d}t \\
&= \int_0^T \|v(t) - (A^*)^{-1}\partial_t v(t)\|_V^2 = \|v - (A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 = \|u\|_{\mathcal{X}}^2 \geq \|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}
\end{aligned}$$

and, therefore,

$$\sup_{w\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(w,v)}{\|w\|_{\mathcal{X}}} \geq \|v\|_{\mathcal{Y}} \quad \forall v\in\mathcal{Y}.$$

This shows the assertion

$$\beta = \beta^* = \inf_{v\in\mathcal{Y}\setminus\{0\}} \sup_{w\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} \geq 1. \qquad \square$$

Summing up these preliminary observations on the inf-sup constants β, $\beta^{(2)}$ and $\widetilde{\beta}^{(2)}$ yields the following explicit bound for $\widetilde{\beta}^{(2)}$ which depends only on the operator norm of the linear operator G_1.

Proposition 4.6. *For $G_1\in\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))$ the inf-sup constant $\widetilde{\beta}^{(2)}$ of the bilinear form $\widetilde{\mathcal{B}}^{(2)}$ satisfies $\widetilde{\beta}^{(2)} \geq 1 - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}$. In particular, the inf-sup condition* (4.2) *is satisfied if*

$$\|G_1\|_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} < 1. \tag{4.8}$$

Proof. Combining the Lemmas 4.2, 4.3 and 4.5 yields the assertion,

$$\widetilde{\beta}^{(2)} \geq \beta^{(2)} - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} \geq \beta^2 - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} \geq 1 - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}.$$

$\square$

To see that Assumption (4.8) is not too restrictive, we calculate $\|G_1\|_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}$ for an explicit example taken from [6].

Example 4.7. Let $U = H$, $Q\in\mathcal{L}_1^+(H)$ and $\mathcal{H} = Q^{1/2}H$. We define the operator $G_1\colon H\to\mathcal{L}_2(\mathcal{H};H)$ by

$$G_1(\varphi)\psi := \sum_{n\in\mathbb{N}} \langle\varphi,\chi_n\rangle_H \langle\psi,\chi_n\rangle_H\, \chi_n, \quad \varphi,\psi\in H,$$

where $(\chi_n)_{n\in\mathbb{N}}\subset H$ is an orthonormal eigenbasis of A with corresponding eigenvalues $(\alpha_n)_{n\in\mathbb{N}}$ (in increasing order). Omitting the calculations we obtain

(i) $G_1\in\mathcal{L}(H;\mathcal{L}_2(\mathcal{H};H))$ with $\|G_1\|_{\mathcal{L}(H;\mathcal{L}_2(\mathcal{H};H))} \leq \operatorname{tr}(Q)^{1/2}$,
(ii) $G_1\in\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};V))$ with $\|G_1\|_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};V))} \leq \operatorname{tr}(Q)^{1/2}$,
(iii) $G_1\in\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))$ with

$$\|G_1\|_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))} \leq \left(\frac{\|Q\|_{\mathcal{L}_2(H;H)}}{\alpha_1}\right)^{1/2}.$$

4.2. **Well-posedness.** Using the result of the previous section on the inf-sup constant of the bilinear form $\widetilde{\mathcal{B}}^{(2)}$ we may apply the Nečas theorem, Theorem 4.1, in order to prove existence and uniqueness of a solution to the deterministic variational problem that we have derived in Section 3 for the second moment of the mild solution.

Theorem 4.8. *Suppose that Assumption* (4.8) *on $G_1\in\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))$ is satisfied. Then the variational problem*

$$w\in\mathcal{X}^{(2)}: \quad \widetilde{\mathcal{B}}^{(2)}(w,v) = {}_{\mathcal{Y}^{(2)*}}\langle f,v\rangle_{\mathcal{Y}^{(2)}} \quad \forall v\in\mathcal{Y}^{(2)} \tag{4.9}$$

admits a unique solution $w\in\mathcal{X}^{(2)}$ for every $f\in\mathcal{Y}^{(2)*}$. In particular, there exists a unique solution $u\in\mathcal{X}^{(2)}$ satisfying (3.9).

Proof. As already mentioned, Theorem 4.1 guarantees existence and uniqueness of a solution $w \in \mathcal{X}^{(2)}$ to (4.9) if the bilinear form $\widetilde{\mathcal{B}}^{(2)}$ satisfies (4.2) and (4.3).

We know from Proposition 4.6 that $\widetilde{\beta}^{(2)} \geq 1 - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}$ and, hence, that $\widetilde{\beta}^{(2)} > 0$ under Assumption (4.8). It remains to show that also the second condition (4.3) is satisfied. For this, fix $v \in \mathcal{Y}^{(2)} \setminus \{0\}$.

$$
\begin{aligned}
\sup_{u \in \mathcal{X}^{(2)}} \widetilde{\mathcal{B}}^{(2)}(u,v) &\geq \sup_{\substack{u \in \mathcal{X}^{(2)} \\ \|u\|_{\mathcal{X}^{(2)}}=1}} \widetilde{\mathcal{B}}^{(2)}(u,v) = \sup_{u \in \mathcal{X}^{(2)} \setminus \{0\}} \frac{\widetilde{\mathcal{B}}^{(2)}(u,v)}{\|u\|_{\mathcal{X}^{(2)}}} \\
&\geq \sup_{u \in \mathcal{X}^{(2)} \setminus \{0\}} \frac{\mathcal{B}^{(2)}(u,v)}{\|u\|_{\mathcal{X}^{(2)}}} - \sup_{w \in \mathcal{X}^{(2)} \setminus \{0\}} \frac{{}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}(\delta((G_1 \otimes G_1)(w)q)), v\rangle_{\mathcal{Y}^{(2)}}}{\|w\|_{\mathcal{X}^{(2)}}} \\
&\geq \beta^{(2)*}\|v\|_{\mathcal{Y}^{(2)}} - \sup_{w \in \mathcal{X}^{(2)} \setminus \{0\}} \frac{\|\delta^*_{\mathcal{Y}}(\delta((G_1 \otimes G_1)(w)q))\|_{\mathcal{Y}^{(2)*}} \|v\|_{\mathcal{Y}^{(2)}}}{\|w\|_{\mathcal{X}^{(2)}}} \\
&\geq \left(\beta^{(2)*} - \|\delta^*_{\mathcal{Y}}(\delta((G_1 \otimes G_1)(\cdot)q))\|_{\mathcal{L}(\mathcal{X}^{(2)};\mathcal{Y}^{(2)*})}\right) \|v\|_{\mathcal{Y}^{(2)}},
\end{aligned}
$$

where we used the reverse triangle inequality and the inf-sup constant $\beta^{(2)*}$ defined in (4.7). Applying Proposition 3.3, Lemma 4.3 and using the fact that $\beta = \beta^*$ yields

$$
\begin{aligned}
\sup_{u \in \mathcal{X}^{(2)}} \widetilde{\mathcal{B}}^{(2)}(u,v) &\geq \left((\beta^*)^2 - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}\right) \|v\|_{\mathcal{Y}^{(2)}} \\
&= \left(\beta^2 - \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_4(\mathcal{H};H))}\right) \|v\|_{\mathcal{Y}^{(2)}} > 0 \qquad \forall v \in \mathcal{Y} \setminus \{0\}
\end{aligned}
$$

under Assumption (4.8) by Lemma 4.5. $\square$

5. From the second moment to the covariance

In the previous sections we have derived a deterministic variational problem satisfied by the second moment $\mathbb{M}^{(2)}X$ of the solution process X to the stochastic partial differential equation (2.2) as well as well-posedness of that problem. In this section we will describe the covariance $\operatorname{Cov}(X)$ also in terms of a deterministic problem. For this purpose, we remark first that

$$
\begin{aligned}
\operatorname{Cov}(X) &= \mathbb{E}\left[(X - \mathbb{E}X) \otimes (X - \mathbb{E}X)\right] \\
&= \mathbb{E}\left[(X \otimes X) - (\mathbb{E}X \otimes X) - (X \otimes \mathbb{E}X) + (\mathbb{E}X \otimes \mathbb{E}X)\right] \\
&= \mathbb{M}^{(2)}X - \mathbb{E}X \otimes \mathbb{E}X
\end{aligned}
$$

and $\operatorname{Cov}(X) \in \mathcal{X}^{(2)}$, since $\mathbb{M}^{(2)}X \in \mathcal{X}^{(2)}$ and $m = \mathbb{E}X \in \mathcal{X}$. By using this relation we are able to show the following result for the covariance $\operatorname{Cov}(X)$ of the mild solution.

Theorem 5.1. *Let all conditions of Assumption 2.1 be satisfied and let X be the mild solution to (2.2). Then the covariance $\operatorname{Cov}(X) \in \mathcal{X}^{(2)}$ solves*

$$
u \in \mathcal{X}^{(2)}: \quad \widetilde{\mathcal{B}}^{(2)}(u,v) = g(v) \quad \forall v \in \mathcal{Y}^{(2)} \tag{5.1}
$$

with $\widetilde{\mathcal{B}}^{(2)}$ as in (3.10),

$$
g(v) := \langle \operatorname{Cov}(X_0), v(0,0)\rangle_{H^{(2)}} + {}_{\mathcal{Y}^{(2)*}}\langle \delta^*_{\mathcal{Y}}(\delta((G(m) \otimes G(m))q)), v\rangle_{\mathcal{Y}^{(2)}}
$$

for $v \in \mathcal{Y}^{(2)}$ and the mean function $m \in \mathcal{X}$ defined in (3.7).

Proof. By the remark above, $\operatorname{Cov}(X) = \mathbb{M}^{(2)}X - \mathbb{E}X \otimes \mathbb{E}X$. By using the result of Theorem 3.5 for the second moment $\mathbb{M}^{(2)}X$ as well as (3.8) for the mean function $m = \mathbb{E}X$ we calculate for $v_1, v_2 \in \mathcal{Y}$:

$$
\begin{aligned}
\widetilde{\mathcal{B}}^{(2)}(\operatorname{Cov}(X), v_1 \otimes v_2) &= \widetilde{\mathcal{B}}^{(2)}(\mathbb{M}^{(2)}X, v_1 \otimes v_2) - \widetilde{\mathcal{B}}^{(2)}(\mathbb{E}X \otimes \mathbb{E}X, v_1 \otimes v_2) \\
&= f(v_1 \otimes v_2) - \mathcal{B}(m, v_1)\,\mathcal{B}(m, v_2) + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*(\delta((G_1(m) \otimes G_1(m))q)), v_1 \otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\
&= \langle \mathbb{M}^{(2)}X_0, (v_1 \otimes v_2)(0,0)\rangle_{H^{(2)}} + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*((G_2 \otimes G_2)q), v_1 \otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\
&\quad + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*((G_1(m) \otimes G_2)q), v_1 \otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\
&\quad + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*((G_2 \otimes G_1(m))q), v_1 \otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\
&\quad - \langle \mathbb{E}X_0, v_1(0)\rangle_H \langle \mathbb{E}X_0, v_2(0)\rangle_H + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*(\delta((G_1(m) \otimes G_1(m))q)), v_1 \otimes v_2\rangle_{\mathcal{Y}^{(2)}} \\
&= \langle \operatorname{Cov}(X_0), (v_1 \otimes v_2)(0,0)\rangle_{H^{(2)}} + {}_{\mathcal{Y}^{(2)*}}\langle \delta_{\mathcal{Y}}^*(\delta((G(m) \otimes G(m))q)), v_1 \otimes v_2\rangle_{\mathcal{Y}^{(2)}}.
\end{aligned}
$$

Hence,

$$
\widetilde{\mathcal{B}}^{(2)}(\operatorname{Cov}(X), v_1 \otimes v_2) = g(v_1 \otimes v_2) \quad \forall v_1, v_2 \in \mathcal{Y}
$$

and the density of the subset $\operatorname{span}\{v_1 \otimes v_2 : v_1, v_2 \in \mathcal{Y}\} \subset \mathcal{Y}^{(2)}$ completes the proof. $\square$

Remark 5.2. Theorem 5.1 shows that if only the covariance of the mild solution to (2.2) needs to be computed, one can do this by solving sequentially two coupled deterministic variational problems: the untensorized problem (3.8) for the mean function and afterwards the tensorized problem (5.1) for the covariance.

Note also that in the case of additive Gaussian noise with a Gaussian initial value the mean and the covariance function already determine the distribution of the solution process uniquely, since it is Gaussian distributed itself.

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