Adaptive Finite Element Method for an Electromagnetic Coefficient Inverse Problem

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Abstract. We present an adaptive finite element method for solution of an electromagnetic coefficient inverse problem to reconstruct the dielectric permittivity and magnetic permeability functions. The inverse problem is formulated as an optimal control problem, where we solve the equations of optimality expressing stationarity of an associated Lagrangian by a quasi-Newton method: in each step we compute the gradient by solving a forward and an adjoint equation. We formulate an adaptive algorithm which can be used to efficiently solve electromagnetic coefficient inverse problem.

Keywords: adaptive finite element method, Maxwell's equations, electromagnetic coefficient inverse problem, ill-posed problems
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INTRODUCTION

We consider an adaptive finite element method for an electromagnetic coefficient inverse problem (CIP) in the form of a parameter identification problem. Our goal is reconstruct dielectric permittivity ε and magnetic permeability μ of the media. We consider the case of a single measurement to recover both coefficients. Inverse scattering is a rapidly expanding area of computational mathematics with a wide range of applications including nondestructive testing of materials, shape reconstruction, non-microscopic ultrasound imaging, subsurface depth imaging of geological structures and seismic prospection.

To solve our inverse problem numerically, we seek to minimize the Tikhonov functional:

$$F(E,\varepsilon,\mu) = \frac{1}{2} \parallel E - \tilde{E} \parallel^2 + \frac{1}{2}\gamma_1 \parallel \varepsilon - \varepsilon_0 \parallel^2 + \frac{1}{2}\gamma_2 \parallel \mu - \mu_0 \parallel^2 . \tag{1}$$

Here E is the vector of the electric field satisfying Maxwell's equations and $\tilde{E}$ is observed data at a finite set of observation points, ε_0 is the initial guess for ε, μ_0 is the initial guess for μ, $\gamma_i, i = 1,2$ are regularization parameters (Tikhonov regularization), and $\parallel \cdot \parallel$ is the discrete L_2 norm.

We formulate the minimization problem as the problem of finding a stationary point of a Lagrangian involving a forward equation (the state equation), a backward equation (the adjoint equation) and an equation expressing that the gradient with respect to the coefficients ε and μ vanishes. To obtain the values of ε and μ we perform an iterative process via solving in each step the forward and backward equations and updating the coefficients ε and μ.

A posteriori error estimate for our CIP can be derived similarly with previous works on this subject [2, 4] and references therein. In this work we use the called all-at-once approach to find Fréchet derivative for the Tikhonov functional. Rigorous derivation of the Fréchet derivatives for state and adjoint problems as well as of the Fréchet derivative of the Tikhonov functional with respect to the coefficient can be done similarly with [5, 6] and will be considered in future work.

The main question of the adaptivity is: *Where to refine the mesh? A posteriori* error analysis answer to this question. In the case of classic forward problems this analysis provides upper estimates for differences between computed and exact solutions locally, in subdomains of the original domain, see, e.g. [1, 7]. However, in the case of inverse problem similar analysis is impossible since every CIP is non-linear and ill-posed. Because of that, an estimate of the difference between computed and exact coefficients is replaced by a posteriori estimate of the accuracy of either the Lagrangian [3] or of the Tikhonov functional [5]. Nevertheless, it was shown in the recent publications [4, 6] that an estimate of the accuracy of the reconstruction of the unknown coefficient is possible in CIPs.

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STATEMENTS OF FORWARD AND INVERSE PROBLEMS

Constrained formulation of Maxwell's equations

We consider the electromagnetic equations in an inhomogeneous isotropic case in the bounded domain $\Omega \subset \mathbb{R}^d,\ d = 2,3$ with boundary $\partial\Omega$. The domain Ω is convex and without reentrant corners, with smooth coefficients ε and μ where value of ε and μ does not varies much. Since we consider practical applications of our method in airport security and imaging of land mines such assumptions are natural. Thus, we are able use the node-based curl-curl formulation with divergence condition of Paulsen and Lynch [9]. Direct application of standard piecewise continuous $[H^1(\Omega)]^3$-conforming FE for the numerical solution of Maxwell's equations can result in spurious solutions. Following [9] we supplement divergence equations for electric and magnetic fields to enforce the divergence condition and reformulate Maxwell equations as a constrained system:

$$\varepsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times E) - s\nabla(\mu^{-1}\nabla \cdot E) = -j, \qquad \nabla \cdot (\varepsilon E) = \rho, \tag{2}$$

and

$$\frac{\partial^2 H}{\partial t^2} + \nabla \times (\varepsilon^{-1} \nabla \times H) - s\nabla(\varepsilon^{-1}\nabla \cdot H) = \nabla \times (\varepsilon^{-1} J), \qquad \nabla \cdot (\mu H) = 0, \tag{3}$$

respectively, where $s > 0$ denotes the penalty factor. Here $E(x,t), H(x,t)$ are the electric and magnetic fields, respectively, while $\varepsilon(x) > 0$ and $\mu(x) > 0$ are the dielectric permittivity and magnetic permeability that depend on $x \in \Omega$, t is the time variable, T is some final time, $J(x,t) \in \mathbb{R}^d$ is a (given) current density, then $j = \frac{\partial J}{\partial t}$, and $\rho(x,t)$ is a given charge density. For simplicity, we consider the system (2) – (3) with homogeneous initial conditions and perfectly conducting boundary conditions.

Statements of forward and inverse problems

In this work as the forward problem we consider Maxwell equation for electric field with homogeneous initial conditions and perfectly conducting boundary conditions

$$\begin{aligned}
\varepsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times E) &- s\nabla(\mu^{-1}\nabla \cdot E) = -j,\ x \in \Omega,\ 0 < t < T, \\
\nabla \cdot (\varepsilon E) &= 0,\ x \in \Omega,\ 0 < t < T, \\
\frac{\partial E}{\partial t}(x,0) = E(x,0) &= 0, \quad \text{in } \Omega, \\
E \times n &= 0, \quad \text{on } \partial\Omega \times (0,T).
\end{aligned} \tag{4}$$

The inverse problem for (3) and appropriate initial and boundary conditions can be formulated similarly and is not considered in this note. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. We assume that the coefficients $\varepsilon(x)$ and μ of equation (4) are such that

$$\varepsilon(x) \in [1, d_1], \mu(x) \in [1, d_2], d_1, d_2 = const. > 1, \tag{5}$$

$$\mu(x), \varepsilon(x) = 1 \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \Omega, \ \ \mu(x), \varepsilon(x) \in C^2\left(\mathbb{R}^3\right). \tag{6}$$

We consider the following

Inverse Problem. *Suppose that the coefficients $\mu(x), \varepsilon(x)$ satisfies (5) and (6), where the numbers $d_1, d_2 > 1$ are given. Assume that the functions $\mu(x), \varepsilon(x)$ are unknown in the domain Ω. Determine the functions $\mu(x), \varepsilon(x)$ for $x \in \Omega$, assuming that the following function $\tilde{E}(x,t)$ is known*

$$E(x,t) = \tilde{E}(x,t), \forall (x,t) \in \partial\Omega \times (0, \infty). \tag{7}$$

TIKHONOV FUNCTIONAL AND OPTIMALITY CONDITIONS

The inverse problem for electromagnetic scattering can be formulated as an optimization problem, where one seek the permittivity $\varepsilon(x)$ and permeability $\mu(x)$, which result in a solution of equation (4) with best least-squares fit to time domain observations E_{obs}, measured at a finite number of observation points. To do so, we seek functions $\varepsilon(x)$ and $\mu(x)$ that minimize the quantity

$$F(E,\varepsilon,\mu) = \frac{1}{2}\int_0^T \int_\Omega (E-\tilde{E})^2 \delta_{obs} dxdt + \frac{1}{2}\gamma_1 \int_\Omega |\varepsilon - \varepsilon_0|^2 \ dx + \frac{1}{2}\gamma_2 \int_\Omega |\mu - \mu_0|^2 \ dx, \tag{8}$$

where $\tilde{E}$ is the observed electric field at x_{obs}, E satisfies the equation (4) and thus depends on ε, μ , $\delta_{obs} = \sum \delta(x_{obs})$ is a sum of multiples of delta-functions $\delta(x_{obs})$ corresponding to the observation points, and $\gamma_1,\ \gamma_2$ are regularization parameters.

To solve this minimization problem we introduce the Lagrangian

$$\begin{aligned} L(u) &= F(E,\varepsilon,\mu) - \int_0^T \int_\Omega \varepsilon \frac{\partial \lambda}{\partial t}\frac{\partial E}{\partial t}\ dxdt + \int_0^T \int_\Omega (\frac{1}{\mu}\nabla \times E)(\nabla \times \lambda)\ dxdt \\ &+ s\int_0^T \int_\Omega (\frac{1}{\mu}\nabla \cdot E)(\nabla \cdot \lambda)\ dxdt + \int_0^T \int_\Omega \nabla \cdot (\varepsilon E)\lambda\ dxdt + \int_0^T \int_\Omega j\lambda\ dxdt, \end{aligned} \tag{9}$$

where $u = (E,\lambda,\varepsilon,\mu)$, and search for a stationary point with respect to u satisfying $\forall \bar{u} = (\bar{E},\bar{\lambda},\bar{\varepsilon},\bar{\mu})$

$$L'(u;\bar{u}) = 0, \tag{10}$$

where $L'(u;\cdot)$ is the Jacobian of L at u. We assume that $\lambda(\cdot,T) = \frac{\partial \lambda}{\partial t}(\cdot,T) = \bar{\lambda}(\cdot,T) = 0$ and $E(\cdot,0) = \frac{\partial E}{\partial t}(\cdot,0) = \bar{E}(\cdot,0) = 0$, together with perfectly conducting boundary conditions $E \times n = \lambda \times n = 0$ and also $n\cdot(\mu^{-1}\nabla\cdot E) = n\cdot E = 0$ on $\partial\Omega$.

The equation (10) expresses that for all $\bar{u}$,

$$\begin{aligned} 0 = \frac{\partial L}{\partial \lambda}(u)(\bar{\lambda}) &= -\int_0^T \int_\Omega \varepsilon \frac{\partial \bar{\lambda}}{\partial t}\frac{\partial E}{\partial t}\ dxdt + \int_0^T \int_\Omega (\frac{1}{\mu}\nabla \times E)(\nabla \times \bar{\lambda})\ dxdt & (11)\\ &+ s\int_0^T \int_\Omega (\frac{1}{\mu}\nabla \cdot E)(\nabla \cdot \bar{\lambda})\ dxdt + \int_0^T \int_\Omega \nabla \cdot (\varepsilon E)\bar{\lambda}\ dxdt + \int_0^T \int_\Omega j\bar{\lambda}\ dxdt, & (12) \end{aligned}$$

$$\begin{aligned} 0 = \frac{\partial L}{\partial E}(u)(\bar{E}) &= \int_0^T \int_\Omega (E-\tilde{E})\ \bar{E}\ \delta_{obs}\ dxdt & (13)\\ &- \int_0^T \int_\Omega \varepsilon \frac{\partial \lambda}{\partial t}\frac{\partial \bar{E}}{\partial t}\ dxdt + \int_0^T \int_\Omega (\frac{1}{\mu}\nabla \times \lambda)(\nabla \times \bar{E})\ dxdt \\ &+ s\int_0^T \int_\Omega (\frac{1}{\mu}\nabla \cdot \lambda)(\nabla \cdot \bar{E})\ dxdt - \int_0^T \int_\Omega \varepsilon \nabla \lambda \bar{E}\ dxdt, & (14) \end{aligned}$$

$$0 = \frac{\partial L}{\partial \varepsilon}(u)(\bar{\varepsilon}) = -\int_0^T \int_\Omega \frac{\partial \lambda}{\partial t}\frac{\partial E}{\partial t}\bar{\varepsilon}\ dxdt - \int_0^T \int_\Omega E\nabla\lambda\bar{\varepsilon}\ dxdt + \gamma_1 \int_\Omega (\varepsilon - \varepsilon_0)\bar{\varepsilon}\ dx,\ x\in\Omega, \tag{15}$$

$$\begin{aligned} 0 = \frac{\partial L}{\partial \mu}(u)(\bar{\mu}) &= -\int_0^T \int_\Omega (\frac{1}{\mu^2}\nabla \times E)(\nabla \times \lambda)\bar{\mu}\ dxdt - s\int_0^T \int_\Omega (\frac{1}{\mu^2}\nabla \cdot E)(\nabla \cdot \lambda)\bar{\mu}\ dxdt & (16)\\ &+ \gamma_2 \int_\Omega (\mu - \mu_0)\bar{\mu}\ dx,\ x\in\Omega. & (17) \end{aligned}$$

The equation (12) is a weak form of the state equation (2), the equation (14) is a weak form of the adjoint state equation

$$\varepsilon \frac{\partial^2 \lambda}{\partial t^2} + \nabla \times (\mu^{-1}\nabla \times \lambda) - s\nabla(\mu^{-1}\nabla \cdot \lambda) = -(E-\tilde{E})\delta_{obs},\ x\in\Omega,\ 0<t<T, \tag{18}$$

$$
\begin{aligned}
\nabla \cdot (\varepsilon\lambda) &= 0, \\
\lambda(\cdot,T) &= \frac{\partial \lambda(\cdot,T)}{\partial t} = 0, \\
\lambda \times n &= 0 \text{ on } \Gamma \times [0,T].
\end{aligned} \tag{19}
$$

Further, (15) and (17) expresses stationarity with respect to ε and μ, correspondingly.

The Adaptive algorithm

The main goal in adaptive error control for the Lagrangian is to find a mesh K_h with as few nodes as possible, such that $||L(u) - L(u_h)|| < tol$, where $tol > 0$ is tolerance chosen by user. Instead of finding $L(u)$ analytically, we can use the a posteriori error estimate similar one obtained in [2]. More precisely, in the computations we shall use the following adaptive algorithm:

Step 0. Choose an initial mesh K_h in Ω and an initial time partition J_0 of the time interval $(0,T)$. Start with the initial approximations $\varepsilon_h^0 = \varepsilon_0$, $\mu_h^0 = \mu_0$ and compute the sequence of ε_h^m, μ_h^m via the following steps:

Step 1. Compute solutions $E_h\left(x,t,\varepsilon_h^m\right)$ and $\lambda_h\left(x,t,\varepsilon_h^m\right)$ of state and adjoint problems of (4) and (18)-(19) on K_h and J_k.

Step 2. Update the coefficients $\varepsilon_h := \varepsilon_h^{m+1}, \mu_h := \mu_h^{m+1}$ on K_h and J_k using the quasi-Newton method, see details in [3]: $\varepsilon_h^{m+1} = \varepsilon_h^m + \alpha_1 g_1^m(x), \mu_h^{m+1} = \mu_h^m + \alpha_2 g_2^m(x)$, where $\alpha_i, i = 1,2$ is step-size in gradient update.

Step 3. Stop computing ε_h^m, μ_h^m and obtain the functions ε_h, μ_h if either $||g_1^m||_{L_2(\Omega)} \leq \theta$, $||g_2^m||_{L_2(\Omega)} \leq \theta$ or norms $||g_1^m||_{L_2(\Omega)}$, $||g_2^m||_{L_2(\Omega)}$ are stabilized. Otherwise set $m := m+1$ and go to step 1. Here θ is the tolerance in quasi-Newton updates.

Step 4. Compute the functions $B_{1,h}(x), B_{2,h}(x)$: $B_{1,h}(x) = \left| \int_0^T \frac{\partial \lambda_h}{\partial t} \frac{\partial E_h}{\partial t} dt + \int_0^T E_h \nabla \lambda_h dt + \gamma_1 \ (\varepsilon_h - \varepsilon_0) \right|, B_{2,h}(x) = \left| \int_0^T (\frac{1}{\mu_h^2} \nabla \times E_h)(\nabla \times \lambda_h)\ dt - s \int_0^T (\frac{1}{\mu_h^2} \nabla \cdot E_h)(\nabla \cdot \lambda_h)\ dt + \gamma_2\ (\mu_h - \mu_0) \right|$. Next, refine the mesh at all points where $B_{1,h}(x) \geq \beta_1 \max_{\overline{\Omega}} B_{1,h}(x)$, $B_{2,h}(x) \geq \beta_2 \max_{\overline{\Omega}} B_{2,h}(x)$. Here the tolerance numbers $\beta_{1,2} \in (0,1)$ are chosen by the user.

Step 5. Construct a new mesh K_h in Ω and a new time partition J_k of the time interval $(0,T)$. On J_k the new time step τ should be chosen in such a way that the CFL condition is satisfied. Interpolate the initial approximations ε_0, μ_0 from the previous mesh to the new mesh. Next, return to step 1 and perform all above steps on the new mesh.

Step 6. Stop mesh refinements if norms defined in step 3 either increase or stabilize, compared with the previous mesh.

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