International Journal of Mathematical Education in Science and Technology, 2014
http://dx.doi.org/10.1080/0020739X.2014.985271

CLASSROOM NOTE

Teaching transforms: a vector approach

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(*Received 10 October 2013*)

This article treats the problem of introducing transform theory (Fourier, Laplace, z) to undergraduate students and we suggest a *vector* approach which means that signals (functions of time) should be treated as vectors from the beginning and that transforms are introduced as a scalar product; the transform should be presented as a tool to analyse the signal exactly in the same way as the dot product is used to analyse an 'arrow' vector in a Cartesian space. Hence, the transform becomes a tool to find the signal's magnitude in the directions of the basis vectors.

Keywords: Fourier transform; vectors; scalar product; basis vectors

1. Introduction

By 'transforms', we refer primarily to Laplace, Fourier and z-transforms, but the arguments and conclusions apply also to wavelet transforms. Even though the arguments are true in general, we will limit the discussion mostly to Fourier transforms of periodic signals and comment on how the results can be applied also to the other, more general transforms, at the end of this article.

The subject of transforms is given as a 'mathematical' subject at some universities and as part of a physics/electrical/mechanical subject at other universities. It is an extremely important analysis tool, most of all for electrical engineers, but also for physicists and mechanical engineers. This is my first conclusion: a transform is an *analysis tool*. It is used to extract information from a signal variation in time (typically); the transform will reveal what (harmonic) frequencies the signal consists of. From conclusion number one, follows immediately conclusion number two; since a transform is an *analysis tool*, university undergraduate courses should focus on how to *use the tool*, i.e. applications.

My personal opinion is that textbooks on transforms (and hence also university courses) focus too much on the mathematical details and not enough on applications and certainly not enough on *understanding* transforms. It is typically considered to be a difficult subject among undergraduates and part of the reason for that is that they do not understand the transform concept nor do they understand how to use it to extract information from a signal.

It is also my opinion that Fourier transforms are not introduced in the best way in most common university textbooks; it is typically introduced as a Fourier series and motivated by a number of examples that illustrate the formulas.

This article suggests a different approach to the introduction of Fourier transforms: *the vector approach*. This suggests that signals should be treated as *vectors* (they *are* vectors) and as vectors they exist in a vector space. Students taking a course on Fourier transforms

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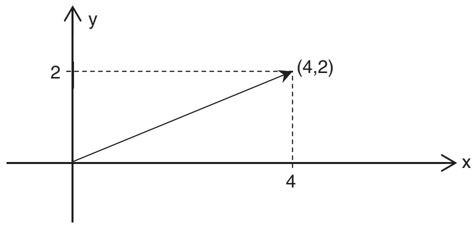

Figure 1. A ‘Cartesian’ vector with coordinates (4,2).

are well familiar with vectors and vector spaces from courses in linear algebra; therefore, this is a natural language for them. Once signals are treated as vectors, the concept of a transform is simply a scalar product that we use to find a signal’s (i.e. a *vector’s*) component along any basis vector.

The rest of this article is organized as follows: In Section 2, we will introduce vectors, vector spaces and basis vectors in general, using the common ‘arrows’ in a Cartesian coordinate system and define the scalar product for these vectors. In Section 3, we will do the same for signals, i.e. a function of time $f(t)$, and demonstrate how naturally the Fourier transform expression appears. Section 4 will illustrate the use of the Fourier transform by some examples and Section 5 discusses the results and how to extend the results to other kinds of signals and transforms. Section 6 concludes the results in this work.

2. Vectors

Vectors are typically introduced as ‘arrows’ in a Cartesian coordinate system as in Figure 1. If we refer to this vector simply as (4,2), most students would have no problem accepting that short vector notation. Students accept these kinds of vectors and the way we refer to them by coordinates only. However, students find it *much* harder accepting that a function, such as the time function in Figure 2, is also a vector and they are *very reluctant* to accept that this vector *can also be represented* by a number of coordinates: $(\ldots,0,0.5j,0,-0.5j,0,0,0\ldots)$. (We will justify this allegation later at the end of Section 4.) In general, vectors like the one in Figure 2 need more than just two coordinates, but then again, so do ‘arrow’ vectors in Figure 1 in an n-dimensional space.

When teaching transforms, it cannot be emphasized enough how important it is to make students accept and treat signals as vectors. We will come back to that later.

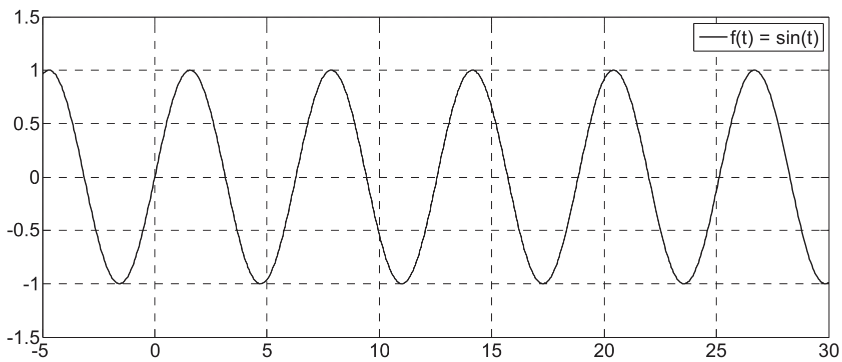

Figure 2. A function of time is also a vector.

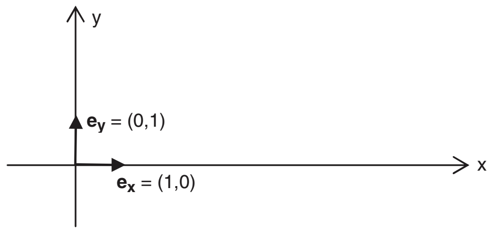

Figure 3. A set of orthogonal basis vectors for the 2D Cartesian vector space.

The (4,2) notation for the 'arrow' vector in Figure 1 is natural for most students but make sure they understand exactly what it means; (4,2) means 4 × (basis vector $\mathbf{e_x}$) + 2 × (basis vector $\mathbf{e_y}$). It is necessary to emphasize the existence and importance of the vector space's set of basis vectors; it is a set of vectors that 'span' the space. Future calculations will be simplified if we can find a set of *orthonormal* basis vectors (i.e. orthogonal and having length = 1). For the vector space in Figure 1, that is elementary, $\mathbf{e_x}$ = (1,0) and $\mathbf{e_y}$ = (0,1) (see Figure 3).

The size (or *length*) of any vector $\bar{v}$ in the direction of a basis vector is the *scalar* product between the vector and the basis vector (the 'dot' product):

$$v_i = \bar{v} \cdot \bar{e}_i \tag{1}$$

For the vector in Figure 1, we get the following components in the *x*- and *y*-directions:

$$v_x = (4, 2) \cdot (1, 0) = 4 \cdot 1 + 2 \cdot 0 = 4 \tag{2}$$

$$v_y = (4, 2) \cdot (0, 1) = 4 \cdot 0 + 2 \cdot 1 = 2 \tag{3}$$

Hence, for any vector, we use the scalar product to find its size along any basis vector. This should be the starting point for an introductory course on transforms.

3. Functions as vectors

There are a certain number of criteria that need to be fulfilled in order to have a vector field.[1] For example, we must be able to define a *zero*vector, and finding a zero vector for functions like the one in Figure 2 is trivial. A less trivial problem is to find a set of basis vectors that span the 'space' of all time functions like the one in Figure 2. Let us consider the 'signal space' and for now we limit the space to include only 'all continuous signals with period *T*'. Figures 4 and 5 illustrate two signals in this space.

Vector theory now suggest the following: since we claim that these signals are *vectors*, we should be able to

(1) find a set of basis vectors that span the space, and
(2) define a scalar product

The fact that a set of basis vectors exists suggests that any vector in the space could be expressed as a linear combination of the basis vectors:

$$\bar{v} = v_1 \cdot \bar{e}_1 + v_2 \cdot \bar{e}_2 + v_3 \cdot \bar{e}_3 + \cdots = \sum_i (v_i \bar{e}_i) = \sum_i (\bar{v} \cdot \bar{e}_i)\, \bar{e}_i \tag{4}$$

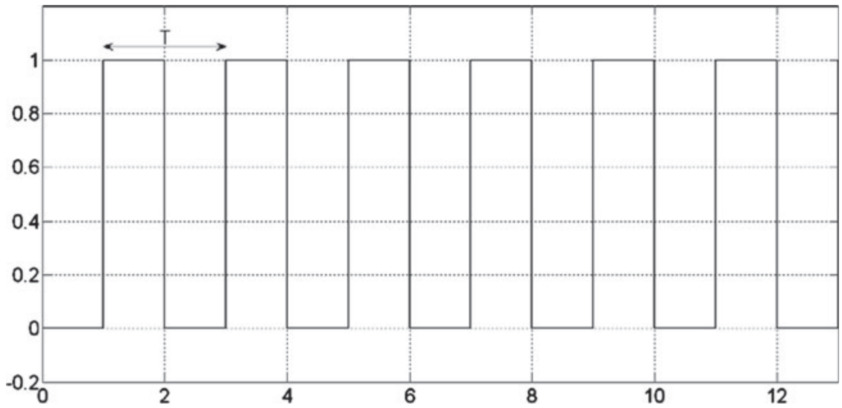

Figure 4. A square signal with period *T*.

The fact that we can define a scalar product gives us a tool to find the coefficients in (4) (as in (2) and (3)). The natural choice of basis vectors would be a set of harmonics; our first choice of basis vectors would be $\cos(\omega_k t + \varphi_k)$, where frequencies ω_k are multiples of $\omega_0 = 2\pi/T$:

$$\omega_k = k \cdot \frac{2\pi}{T} = k \cdot \omega_0, k = 0, 1, 2, \ldots \tag{5}$$

This is appealing since it suggests that any signal in the signal space could be expressed as

$$v(t) = a_0 \cdot 1 + a_1 \cdot \cos(1 \cdot \omega_0 t + \varphi_1) + a_2 \cdot \cos(2 \cdot \omega_0 t + \varphi_2) + \cdots. \tag{6}$$

This would work, but we have two small problems: each basis vector has two parameters (k and φ_k) and the basis vectors are not orthonormal (let students prove that). However, we can solve these problems by choosing our basis vectors slightly differently; we will instead choose $e^{jk\omega_0 t}$. Note that due to Euler's formula for cosine this is only a minor change of

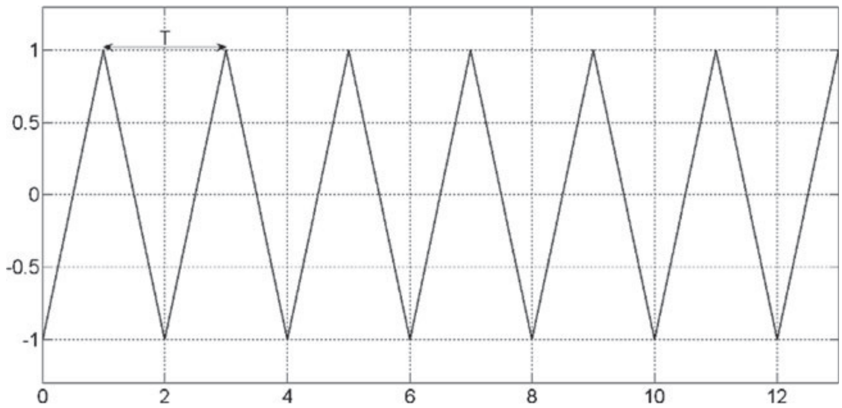

Figure 5. A triangular signal with period *T*.

basis vectors:

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2} \tag{7}$$

As a matter of fact, if we extend the domain of k to include also negative values and disregard the $\frac{1}{2}$ factor, we have not changed anything. (In Appendix 1, we prove that $\{e^{jk\omega_0 t}\}$ is indeed a set of basis vectors.) Hence, with our new set of basis vectors, we should be able to express any signal in our space as

$$v(t) = \ldots c_{-1} \cdot e^{-j\omega_0 t} + c_0 + c_1 \cdot e^{j\omega_0 t} + c_2 \cdot e^{j2\omega_0 t} + \cdots. \tag{8}$$

or, simply

$$v(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega_0 t} \tag{9}$$

This is a nice and compact expression. We need to find the coefficients c_k of course, but from vector theory we know that that is only a matter of taking the scalar product between the vector and the basis vectors. We will do that but first let us try to figure out what they will look like. If we compare expressions (9) and (6), we can see that the phase information is no longer in the basis vector; the c_k coefficients must contain information about amplitude *and phase* for each cosine in (6). The only way to do that is if the c_k coefficients are complex numbers, the cosine's amplitude is in the complex numbers' magnitude and the cosine's phase angle is in the complex numbers' argument. We will relate the c_k coefficients to the a_k and φ_k parameters later.

In order to find the c_k coefficients, we need to define the scalar product between two signals. The scalar product between any two functions is in general defined as follows:[2]

$$\int f(t) \cdot g^*(t)\mathrm{d}t \tag{10}$$

where g^* represents the complex conjugate of g. For two signals with period T, this expression changes to

$$\frac{1}{T}\int_0^T f(t) \cdot g^*(t)\,\mathrm{d}t \tag{11}$$

In order to find out the size of our signal $v(t)$ in the direction of any basis vector $e^{jk\omega_0}$, we apply expression (11) and so the size is represented by the coefficient c_k:

$$c_k = \frac{1}{T}\int_0^T v(t) \cdot e^{-jk\omega_0 t}\mathrm{d}t = V(\omega) = V(k\omega_0) = V(k) \tag{12}$$

Expression (12) is the Fourier transform of the signal $v(t)$ as it appears in most textbooks, and it is typically denoted by $V(\omega)$ or $V(k)$, but it is important (*very important*!) to understand that it is just a scalar product that we use to find the signal's size in the direction of each basis vector. In general, it has to be complex numbers since it must carry information about both the amplitudes and phase angles of the cosines in expression (6).

The c_k coefficients really carry all the information we need about the signal $v(t)$, however, for pedagogical reasons, I stress the importance of relating the c_k coefficients to the a and φ parameters in expression (6) in undergraduate courses. That really helps making the Fourier transform's output understandable. The relationship between the c_k coefficients and a and φ parameters is straightforward:

$$\begin{aligned} a_k \cos(k\omega_0 t + \varphi_k) &= a_k \frac{\mathrm{e}^{j(k\omega_0 t + \varphi_k)} + \mathrm{e}^{-j(k\omega_0 t + \varphi_k)}}{2} \\ &= \frac{1}{2} a_k \left(\mathrm{e}^{jk\omega_0 t} \cdot \mathrm{e}^{j\varphi_k} + \mathrm{e}^{-jk\omega_0 t} \cdot \mathrm{e}^{-j\varphi_k} \right) \\ &= \underbrace{\frac{1}{2} a_k \cdot \mathrm{e}^{j\varphi_k}}_{c_k} \cdot \mathrm{e}^{jk\omega_0 t} + \underbrace{\frac{1}{2} a_k \cdot \mathrm{e}^{-j\varphi_k}}_{c_k^*} \cdot \mathrm{e}^{-jk\omega_0 t} \\ &= c_k \cdot \mathrm{e}^{jk\omega_0 t} + c_k^* \cdot \mathrm{e}^{-jk\omega_0 t} \end{aligned}$$

$$\Rightarrow c_k = \frac{1}{2} a_k \cdot \mathrm{e}^{j\varphi_k} \tag{13}$$

$$\Rightarrow a_k = 2 \cdot |c_k| \tag{14}$$

$$\varphi_k = \arg\ c_k \tag{15}$$

Expressions (14) and (15) relate the Fourier transform output to the linear combination of cosines in expression (6). These expressions are valid for $k \neq 0$. The special case $k = 0$ should be treated separately. $k = 0$ corresponds to a frequency $= 0$, i.e. the signal's DC offset which equals a_0 in expression (6). A signal's DC offset is simply its average value over one period:

$$a_0 = \frac{1}{T} \int_0^T v(t)\ \mathrm{d}t \tag{16}$$

Inserting $k = 0$ into (12) gives us

$$c_0 = \frac{1}{T} \int_0^T v(t) \cdot e^{-j0}\ \mathrm{d}t = \frac{1}{T} \int_0^T v(t)\ \mathrm{d}t \tag{17}$$

Since (17) and (16) are the same, we conclude that for $k = 0$,

$$c_0 = a_0 \tag{18}$$

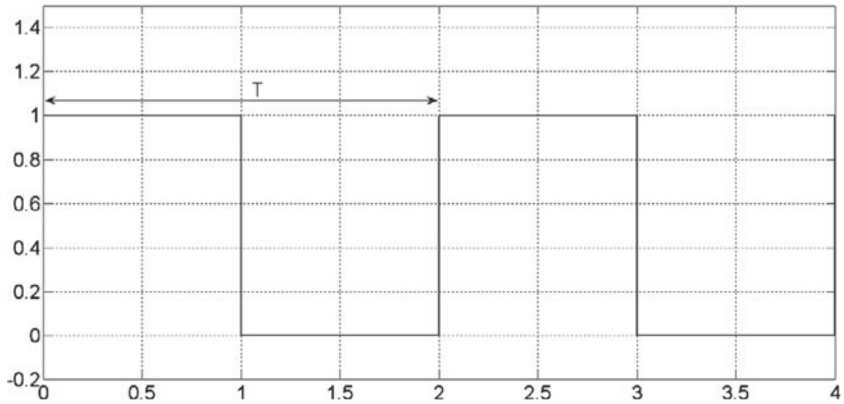

Figure 6. Try to *guess* the first few Fourier coefficients of this square signal.

4. Examples

Consider the signal in Figure 6.

In a typical undergraduate course on Fourier transforms, this is one of the first signals that a student is asked to transform. This is how we suggest that this problem should be approached.

First, in order to encourage the *understanding* of Fourier transforms and Fourier coefficients, ask the students to *guess* the values of the first few Fourier coefficients (three or four: c_0, c_1, c_2, c_3). Once you understand the relationship between the c_k coefficients and a and φ parameters, it is not that difficult. The following way of reasoning should be encouraged. First of all, the signal's DC offset is 0.5 (volts?) and hence I expect to get $c_0 = a_0 = 0.5$; the first Fourier coefficient was simple enough. The Fourier coefficient c_1 corresponds to a cosine of frequency $\omega = 1{\cdot}\omega_0 = 2\pi/T$, and this cosine has been plotted in the same graph as the signal $v(t)$ in Figure 7.

Look at expression (6); we should add a number of cosines so that it becomes a square wave. Intuitively, we can see that it will require a 'lot' of the basis vector with index $k = 1$;

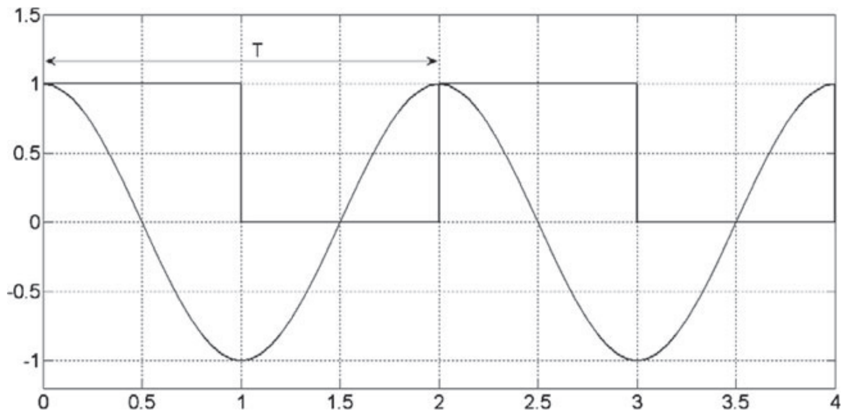

Figure 7. We have added the basis vector corresponding to $k = 1$.

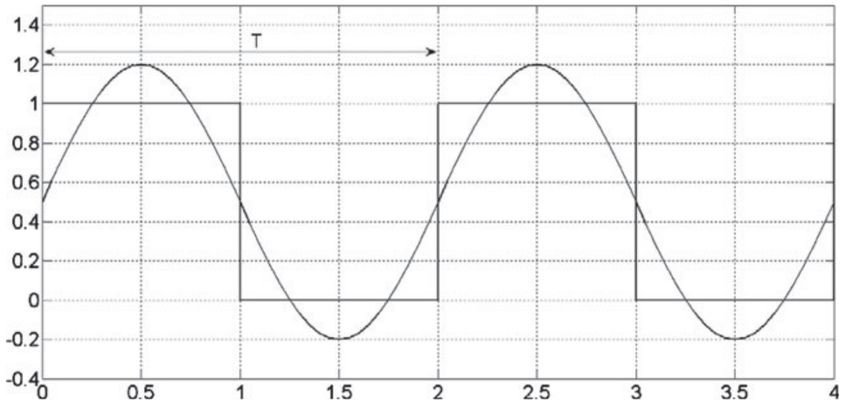

Figure 8. We have added the basis vector corresponding to $k = 1$ and $k = 0$.

if we first add a phase shift of $-90°$ and reduce the amplitude a little, we get Figure 8. (In Figure 8, we have also added the DC offset a_0 that we determined above.)

Look at Figure 8. We have only added the signals of our two first guesses and we already have something that resembles the signal $v(t)$, or at least, we can see that we are on the right track. The amplitude of the cosine in Figure 8 is approximately 0.7 and the cosine has been phase shifted $-90°$. From expression (13), we 'guess' that the Fourier coefficient c_1 will be approximately

$$c_1 \approx \frac{1}{2} \cdot 0.7 \cdot e^{-j\pi/2} = -0.35j \tag{19}$$

Let us 'guess' some more. Look at Figure 8. The signal $v(t)$ is odd (if we subtract the DC offset), i.e. $v(t) = -v(-t)$. A linear combination of signals that should represent $v(t)$ must then necessarily also be odd; all components in expression (6) must have a phase shift of either $+\pi/2$ or $-\pi/2$. In other words, all Fourier coefficients will be imaginary (the real part will always be zero) as in expression (19). For example, the component corresponding to $k = 2$ is the signal $\cos(2 \cdot \omega_0 t \pm \pi/2)$. In Figure 9, we have added a small part of this signal to the cosine in Figure 8 (for a phase shift of $-\pi/2$).

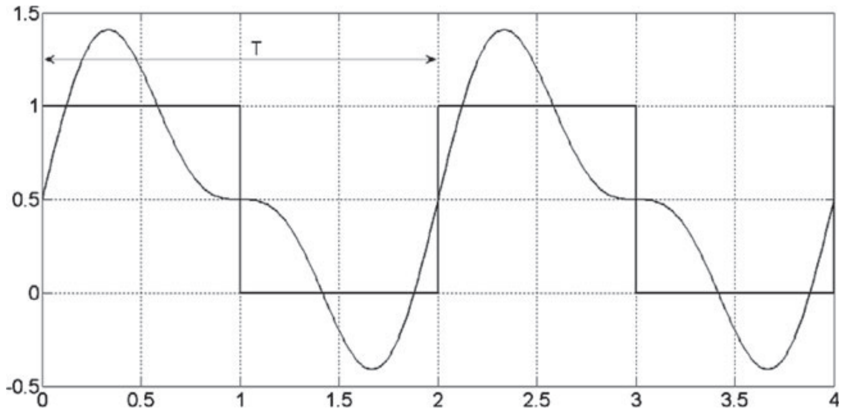

Figure 9. We have added the basis vector corresponding to $k = 2$; does not seem to fit.

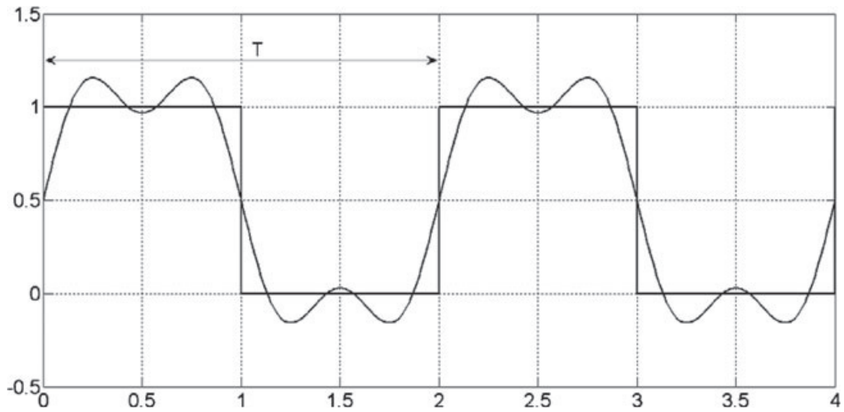

Figure 10. We have added the basis vector corresponding to $k = 3$; seems to fit very well.

It is hard to argue that this made things better and we conclude that the signal's size in the direction of the basis vector with $k = 2$ is probably very small and hence we predict that

$$c_2 \approx 0 \tag{20}$$

In Figure 10, we have instead added a small fraction of the basis vector representing $k = 3$ (and phase shifted $-\pi/2$).

This does make the added signals look more like $v(t)$, and since the amplitude of the cosine we added was 0.25, we guess that

$$c_3 \approx \frac{1}{2} \cdot 0.25 \cdot \mathrm{e}^{-j\pi/2} = -0.125j \tag{21}$$

I personally think that the guesswork we have done above is the most important part when learning Fourier transform theory since it very clearly relates the Fourier transform output to the signal information that we are looking for.

When we are done guessing, we verify our guesses by finding the exact Fourier transform coefficients by using expressions (12) and (17). Expression (17) will give us $a_0 = c_0$:

$$c_0 = \frac{1}{T}\int_0^T v(t)\ \mathrm{d}t = \frac{1}{T}\int_0^{T/2} 1\ \mathrm{d}t = \frac{1}{T}\,[t]_0^{T/2} = \frac{1}{T}\left\{\frac{T}{2} - 0\right\} = \frac{1}{2} = a_0 \tag{22}$$

which agrees exactly with our previous guess. Next, we try to find a general expression for c_k, $k \neq 0$. Expression (12) gives us

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T v(t)\cdot \mathrm{e}^{-jk\omega_0 t} = \frac{1}{T}\int_0^{T/2} 1\cdot \mathrm{e}^{-jk\omega_0 t}\mathrm{d}t = \frac{1}{-jk\omega_0 T}\left[\mathrm{e}^{-jk\omega_0 t}\right]_0^{T/2} \\ &= \frac{1}{-jk2\pi}\left(\mathrm{e}^{-jk\pi} - 1\right) \end{aligned} \tag{23}$$

where we have used the relationship $\omega_0 T = 2\pi$. In expression (23), we first extract $e^{-jk\pi/2}$ from the parentheses, and then apply Euler's formula for sine:

$$c_k = \frac{1}{-jk2\pi}\left(e^{-jk\pi/2} - e^{jk\pi/2}\right)\cdot e^{-jk\pi/2} = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)\cdot e^{-jk\pi/2} \tag{24}$$

From expression (24), we can see that $c_k = 0$ whenever k is an even number (which agrees very well with our guess that $c_2 \approx 0$). We can also see that all the phase angles will be $\pm\pi/2$, which also agrees with our previous guess. For the odd k's, we have

$$c_1 = \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right)\cdot e^{-j\pi/2} = \frac{1}{\pi}\cdot 1\cdot(-j) \approx -0.318j \tag{25}$$

(compare with (19))

$$c_3 = \frac{1}{3\pi}\sin\left(\frac{3\pi}{2}\right)\cdot e^{-j3\pi/2} = \frac{1}{3\pi}\cdot(-1)\cdot j \approx -0.106j \tag{26}$$

(compare with (21))

$$c_5 = \frac{1}{5\pi}\sin\left(\frac{5\pi}{2}\right)\cdot e^{-j5\pi/2} = \frac{1}{5\pi}\cdot 1\cdot(-j) \approx -0.063j \tag{27}$$

and so on. Next, we use expressions (14) and (15) to get the amplitudes and phase angles of the corresponding cosines:

$$a_1 = 2\cdot 0.318 = 0.636,\quad a_3 = 2\cdot 0.106 = 0.212,\quad a_5 = 2\cdot 0.063 = 0.126,\quad \text{etc.}$$

$$\varphi_1 = -\frac{\pi}{2},\quad \varphi_3 = -\frac{\pi}{2},\quad \varphi_5 = -\frac{\pi}{2},\quad \text{etc.}$$

In Figure 11, we have plotted expression (6) for $k = 0$–5.

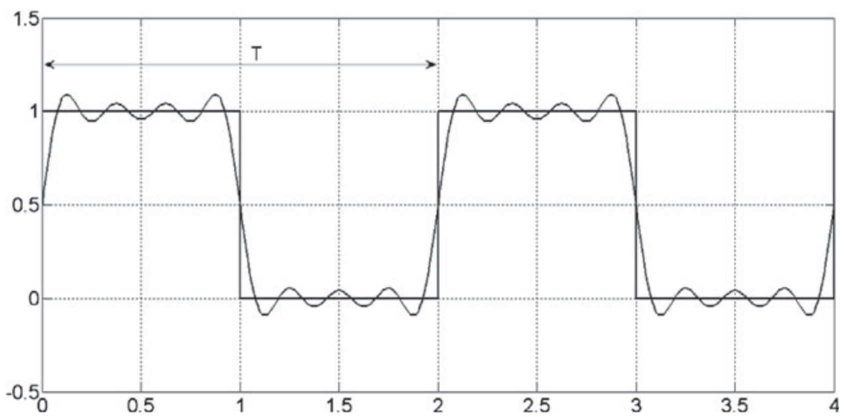

Figure 11. We have approximated the square wave with the first five terms in expression (6).

In Section 2, we suggested that the signal sin $\omega_0 t$ could be represented by the vector coefficients (. . .,0,0,0,0.5j,0,−0.5j,0,0,0,. . ...). We will now justify that allegation. If we insert $v(t) = \sin \omega_0 t$ into expression (12), we get (remember that $\omega_0 T = 2\pi$) the following:

$$c_k = \frac{1}{T}\int_0^T \sin\omega_0 t \times \mathrm{e}^{-jk\omega_0 t}\mathrm{d}t = \left\{\begin{matrix}\text{Using Euler}\,'\text{s}\\ \text{formula for sine}\end{matrix}\right\}$$

$$= \frac{1}{2jT}\int_0^T \left(\mathrm{e}^{j\omega_0 t} - \mathrm{e}^{-j\omega_0 t}\right) \times \mathrm{e}^{-jk\omega_0 t}\mathrm{d}t$$

$$= \frac{1}{2jT}\int_0^T \left(\mathrm{e}^{j(1-k)\omega_0 t} - \mathrm{e}^{-j(1+k)\omega_0 t}\right)\mathrm{d}t \tag{28}$$

$$= \frac{1}{2jT}\left[\frac{\mathrm{e}^{j(1-k)\omega_0 t}}{j\,(1-k)\,\omega_0} - \frac{\mathrm{e}^{-j(1+k)\omega_0 t}}{-j\,(1+k)\,\omega_0}\right]_0^T$$

$$= \frac{1}{2jT} \times \frac{1}{j\omega_0}\left(\frac{\mathrm{e}^{j(1-k)2\pi} - 1}{(1-k)} + \frac{\mathrm{e}^{-j(1+k)2\pi} - 1}{j\,(1+k)}\right) \tag{29}$$

Expression (29) will be equal to 0 for any $k \neq \pm 1$, since the numerators in the parentheses will always be 0. Hence, $c_k = 0$, if $k \neq \pm 1$. If we set $k = +1$ in expression (28), we get

$$c_1 = \frac{1}{2jT}\int_0^T \left(1 - \mathrm{e}^{-j2\omega_0 t}\right)\mathrm{d}t = \frac{1}{2jT}\left[t + \frac{\mathrm{e}^{-j2\omega_0 t}}{j2\omega_0}\right]_0^T$$

$$= \frac{1}{2jT}\left(T + \frac{\mathrm{e}^{-j4\pi}}{j2\omega_0} - 0 - \frac{1}{j2\omega_0}\right) = \frac{1}{2j} = -0.5j \tag{30}$$

In the same way, we would get $c_{-1} = 0.5j$. Consequently, if we assume that the basis vectors are known to be the complex exponentials in expression (8), we could specify our vector sin $\omega_0 t$ using only the vector coefficients as follows:

$$\sin\omega_0 t = \left(\ldots., 0, 0, 0, \overbrace{0.5j}^{k=-1}, \underbrace{0}_{k=0}, \overbrace{-0.5j}^{k=1}, 0, 0, 0, 0\ldots.\right) = (0.5j, 0, -0.5j) \tag{31}$$

5. Discussion

At the beginning we pointed out that ‘arrow’ vectors in a Cartesian coordinate system can be referred to by coordinates like (4,2) as in Figure 1. This notation works because it is well understood that we use a set of orthonormal basis vectors {(1,0) and (0,1)}, and (4,2) simply refers to the vector’s magnitude in the two directions of the basis vectors. It is a convenient and compact way to represent vectors. Notice that the *order* in which the coordinates are given is important; it is understood that the x-coordinate comes first and the y-coordinate is the second ((4,2) $\neq$ (2,4)).

It is less common to represent ‘signal’ vectors with the same compact notation, but this work has pointed out that possibility as a means to teach transforms. We can refer to the 'arrow' vector in Figure 1 by only specifying coefficients (4,2) and the reason is that the basis vectors are 'obvious' and/or understood. We can do that with *any* vectors! The difference is only that the set of basis vectors is less 'obvious'/common and that the coefficients may not necessarily be real numbers. If the basis vectors were widely understood, we could absolutely refer to the signal in Figure 6 as only $(\ldots, 0.63j, 0.126j, 0.318j, 0.5, -0.318j, -0.126j, -0.063j\ldots\ldots)$. (Notice again that the *order* is important.)

This work was limited to signal vectors with period T, and we saw that it requires infinitely many (but countable) basis vectors. For non-periodic signals, the range of basis vectors will be a continuum and referring to such a signal using only the transform coefficients is meaningless; instead, we typically plot $|V(\omega)|$ and $\varphi(\omega)$ as a function of ω.

6. Conclusions

I cannot stress enough the importance of the guesswork we did above. A student who can guess the approximate values of the Fourier coefficients really understand the Fourier transform and the meaning of the Fourier transform output. When you have reached that level of understanding, the Fourier transform becomes the useful *tool* it is supposed to be. Honestly, a student who is able to find the exact values of Fourier transform coefficients by applying expression (12) does not necessarily understand any transform theory at all; all it proves it that he/she can integrate a complex function (and a talented high-school student can do that). Yet, undergraduate courses are very focused on solving expression (12) for more and more complex signals and in my opinion, this attitude needs to change, both among textbook writers and university teachers; focus should primarily be on the understanding of transforms rather than the calculus. Computers can help a student do the math; it is the understanding of the transform output that is the hard part.

Our arguments above would apply to Laplace and z-transforms too, but we would have other sets of basis vectors. For example, the Laplace transform requires that we also include basis vectors with exponentially growing/decaying amplitudes.

I have used this approach in transform teaching for years and my experience is that the vector approach gives student a whole new level of the understanding of transforms. In my classes, I also strongly emphasize the 'guesswork' we did in Section 4. This tells me whether or not a student has understood the transform concept. Once you understand that, you can start calculating transform expressions like the one in (12), but I argue that these calculations are not very helpful/useful unless you have first reached a level of understanding where you can explain exactly what the complex numbers produced by the transform expression represent.

References

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Appendix 1. Proving that $\{e^{jk\omega_0 t}\}_{k=-\infty}^{\infty}$ is a set of basis vectors

In order to prove that $\{e^{jk\omega_0 t}\}_{k=-\infty}^{\infty}$ is indeed a set of basis vectors, we must prove that

(1) all $e^{jk\omega_0 t}$ are linearly independent, and
(2) $\{e^{jk\omega_0 t}\}_{k=-\infty}^{\infty}$ span the space

The set of basis vectors span the space if each vector in the space can be represented by one and only one unique linear combination of the basis vectors.

We will simplify prove by only treating the case where $k = 1$, 2 and 3, i.e.

$$\{e^{jk\omega_0 t}\}_1^3 = \left(e^{j\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}\right)$$

The proof is still general and can be applied to any dimension of $\{e^{jk\omega_0 t}\}$. We will assume that $\omega_0 \neq 0$ (because if $\omega_0 = 0$, the signal is a trivial DC signal).

Proof of linear independency

The set of basis vectors $\left(e^{j\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}\right)$is (by definition) linearly independent, if

$$\lambda_1 \times e^{j\omega_0 t} + \lambda_2 \times e^{j2\omega_0 t} + \lambda_3 \times e^{j3\omega_0 t} = 0 \tag{A.1}$$

has the only solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. We can rewrite (A.1) as

$$\underbrace{e^{j\omega_0 t}}_{\neq 0} \times \left(\lambda_1 + \lambda_2 \times e^{j\omega_0 t} + \lambda_3 \times e^{j2\omega_0 t}\right) = 0 \tag{A.2}$$

(assuming $t \geq \infty$). In (A.2), the second factor must be 0; therefore,

$$\lambda_1 + \lambda_2 \times e^{j\omega_0 t} + \lambda_3 \times e^{j2\omega_0 t} = 0 \tag{A.3}$$

Next, we take the derivative of both sides (with respect to t),

$$j\omega_0\lambda_2 \times e^{j\omega_0 t} + j2\omega_0\lambda_3 \times e^{j2\omega_0 t} = 0$$

$$\underbrace{j\omega_0 \times e^{j\omega_0 t}}_{/} = \times \left(\lambda_2 + 2\lambda_3 \times e^{j\omega_0 t}\right) = 0$$

(since $\omega_0 \neq 0$ and if $t \geq \infty$.) Again, the second factor must be 0; therefore,

$$\lambda_2 + 2\lambda_3 \times e^{j\omega_0 t} = 0 \tag{A.4}$$

Differentiating one more time gives us

$$j\omega_0 \times 2\lambda_3 \times e^{j\omega_0 t} = 0 \tag{A.5}$$

Equation (A.5) is true only if $\lambda_3 = 0$. This indicates that $\lambda_2 = 0$ in (A.4), and then λ_1 in (A.3) must also be 0 and we have proven that expression (A.1) is true only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and hence the vectors $\left(e^{j\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}\right)$ are linearly independent.

Proof of uniqueness

We must also prove that each vector representation is unique. Assume, therefore, that the same vector $u(t)$ can be represented by two sets of coefficients, c_k and d_k:

$$u(t) = c_1 \times e^{j\omega_0 t} + c_2 \times e^{j2\omega_0 t} + c_3 \times e^{j3\omega_0 t} = d_1 \times e^{j\omega_0 t} + d_2 \times e^{j2\omega_0 t} + d_3 \times e^{j3\omega_0 t} \tag{A.6}$$

Then, we must have that

$$(c_1 - d_1) \times e^{j\omega_0 t} + (c_2 - d_2) \times e^{j2\omega_0 t} + (c_3 - d_3) \times e^{j3\omega_0 t} = 0 \qquad \text{(A.7)}$$

However, since we know that the vectors are linearly independent, the only solution for (A.7) is that

$$\left.\begin{array}{l} c_1 - d_1 = 0 \\ c_2 - d_2 = 0 \\ c_3 - d_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} C_1 = d_1 \\ C_2 = d_2 \\ C_3 = d_3 \end{array} \qquad \text{(A.8)}$$

which proves that there is only one unique representation of each vector in the space spanned by $\left(e^{j\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}\right)$, and this proof is also valid for the general case $\left\{e^{jk\omega_0 t}\right\}_{k=-\infty}^{\infty}$.

Proof of orthonormality

In order to have a 'good' set of basis vectors, we also want them to be 'orthonormal', i.e. orthogonal and having length equal to 1. Our choice of basis vectors is indeed orthonormal and we prove that by taking the scalar product of two arbitrary basis vectors (remember that two vectors are orthogonal if their scalar product = 0),

$$\left\langle e^{jn\omega_0 t}, e^{jm\omega_0 t}\right\rangle = \frac{1}{T}\int_0^T e^{jn\omega_0 t} \times e^{jm\omega_0 t}\,\mathrm{d}t = \frac{1}{T}\int_0^T e^{j(n-m)\omega_0 t}\,\mathrm{d}t$$

$$= \frac{1}{T} \times \frac{1}{j(n-m)\omega_0} \times \left[e^{j(n-m)\omega_0 t}\right]_0^T \qquad \text{(A.9)}$$

$$= \frac{1}{j2\pi(n-m)} \times \left(e^{j(n-m)2\pi} - 1\right) = 0 \text{ if } n \neq m \qquad \text{(A.10)}$$

Equation (A.10) proves that our basis vectors are all orthogonal. The 'length' of a vector is the square root of the scalar product between the vector and itself:

$$\left\| e^{jn\omega_0 t} \right\| = \sqrt{\left\langle e^{jn\omega_0 t}, e^{jn\omega_0 t}\right\rangle} = \left\{ \begin{array}{l} \text{set } n = m \\ \text{in (A.9)} \end{array} \right\} = \sqrt{\frac{1}{T}\int_0^T 1\mathrm{d}t} = \sqrt{\frac{1}{T} \times T} = 1 \qquad \text{(A.11)}$$

Hence, our choice of basis vectors, $\left\{e^{jk\omega_0 t}\right\}_{k=-\infty}^{\infty}$, is linearly independent, and they span the space and they are an orthonormal set of basis vectors, i.e. simply a very good choice of basis vectors.