VARIOUS APPROACHES TO PRODUCTS OF RESIDUE CURRENTS

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ABSTRACT. We describe various approaches to Coleff-Herrera products of residue currents R^j (of Cauchy-Fantappiè-Leray type) associated to holomorphic mappings f_j. More precisely, we study to which extent (exterior) products of natural regularizations of the individual currents R^j yield regularizations of the corresponding Coleff-Herrera products. Our results hold globally on an arbitrary pure-dimensional complex space.

1. INTRODUCTION

Let f be a holomorphic function defined on the unit ball $\mathbb{B} \subset \mathbb{C}^n$. If f is a monomial it is elementary to show, e.g., by integrations by parts or by a Taylor expansion, that the principal value current $\varphi \mapsto \lim_{\epsilon\to 0}\int_{|f|^2>\epsilon}\varphi/f$, $\varphi \in \mathscr{D}_{n,n}(\mathbb{B})$, exists and defines a $(0,0)$-current $1/f$ that we also denote by U^f. From Hironaka's theorem it then follows that such limits exist for general f and also that $\mathbb{B}$ may be replaced by a complex space, [20]. The $\bar\partial$-image, $R^f := \bar\partial(1/f)$, is the residue current of f and by Stokes' theorem it is given by $\varphi \mapsto \lim_{\epsilon\to 0}\int_{|f|^2=\epsilon}\varphi/f$, $\varphi \in \mathscr{D}_{n,n-1}(\mathbb{B})$. It has the useful property that its annihilator ideal is equal to the principal ideal $\langle f\rangle$ and, moreover, it gives a factorization of Lelong's integration current; $2\pi i[f=0] = df \wedge \bar\partial(1/f)$.

There are (at least) two natural ways of regularizing U^f and R^f. If $\lambda \in \mathbb{C}$ and $\mathfrak{Re}\,\lambda \gg 0$, then $\lambda \mapsto \int \varphi\,|f|^{2\lambda}/f$ is holomorphic for any test form φ. It is well known (cf., Lemma 6) that the current-valued map $\lambda \mapsto |f|^{2\lambda}/f =: U^{f,\lambda}$ has a meromorphic extension to $\mathbb{C}$ with poles contained in the set of negative rational numbers and that the value at $\lambda = 0$ is U^f. It follows that $\lambda \mapsto \bar\partial|f|^{2\lambda}/f =: R^{f,\lambda}$ is meromorphic in $\mathbb{C}$, analytic in a half space containing the origin, and that the value at the origin is R^f. The technique of using analytic continuation in residue current theory has its roots in the work of Atiyah, [8], and Bernstein-Gel'fand, [14]. In the context of residue currents it has been developed by several authors, e.g., Barlet-Maire, [9], Yger, [33], Berenstein-Gay-Yger,[11], Passare-Tsikh, [27], and recently by the second author in [30]. The second regularization method, inspired by Passare, [23], is more

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explicit and concrete; U^f and R^f are obtained as weak limits of explicit smooth forms. Let χ be a smooth regularization of the characteristic function $\mathbf{1}_{[1,\infty)}$ and let $U^{f,\epsilon} := \chi(|f|^2/\epsilon)/f$ and $R^{f,\epsilon} := \bar\partial\chi(|f|^2/\epsilon)/f$. Then (see, e.g., [23]) $U^f = \lim_{\epsilon\to 0^+} U^{f,\epsilon}$ and $R^f = \lim_{\epsilon\to 0^+} R^{f,\epsilon}$ in the sense of currents. Notice that the original definition mentioned above corresponds to $\chi = \mathbf{1}_{[1,\infty)}$.

If f is a tuple of functions or a section of a vector bundle there are natural analogues of the currents $1/f$ and $\bar\partial(1/f)$ introduced in [28] and [1]. The construction of these more general currents, still denoted U^f and R^f, is based on Bochner-Martinelli and Cauchy-Fantappié-Leray type formulas; see Section 2 for details. In this paper we consider products of regularized currents of this kind and we investigate their limit behavior. It turns out that both the λ-approach and the ϵ-approach yield the same current as the classical Coleff-Herrera approach.

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Let Z be a reduced complex space of pure dimension n, let $E_1, \dots, E_p$ be hermitian holomorphic vector bundles over Z, and let f_j be a holomorphic section of E_j^*. Then $U^{f_j} =: U^j$ and $R^{f_j} =: R^j$ become currents with values in ΛE_j; if $\operatorname{rank} E_j = 1$ then U^j is the principal value current associated with the meromorphic section $1/f_j$ of E_j and $R^j = \bar\partial U^j$. In complete analogy with the regularization methods discussed above we have

$$U^j = U^{j,\lambda}\big|_{\lambda=0} = \lim_{\epsilon\to 0^+} U^{j,\epsilon} \quad \text{and} \quad R^j = R^{j,\lambda}\big|_{\lambda=0} = \lim_{\epsilon\to 0^+} R^{j,\epsilon},$$

see Section 2. We define products of the R^j (for simplicity we restrict attention to such products in this section) recursively as follows: Having defined $R^{k-1}\wedge\cdots\wedge R^1$ it turns out (see [7] or Section 2) that

$$\lambda \mapsto R^{k,\lambda}\wedge R^{k-1}\wedge\cdots\wedge R^1$$

has an analytic continuation to a neighborhood of $\lambda = 0$ and we define $R^k\wedge\cdots\wedge R^1$ as the value at $\lambda = 0$. From the proof of Proposition 5.4 in [6] it follows that one can compute the product in the following way: If $a_1 > \cdots > a_p > 0$ are integers then

$$R^p\wedge\cdots\wedge R^1 = R^{p,\lambda^{a_p}}\wedge\cdots\wedge R^{1,\lambda^{a_1}}\big|_{\lambda=0}.$$

That is, the recursive definition can be replaced by the evaluation of a one-variable analytic (current valued) function at the origin; we just have to make sure that λ^{a_1} tends to zero much faster than λ^{a_2} and so on.

We now consider the smooth form $R^{p,\epsilon_p}\wedge\cdots\wedge R^{1,\epsilon_1}$ and limits of it of the following kind:

Definition 1. Let ϑ be a function defined on $(0,\infty)^p$. We let

$$\lim_{\epsilon_1\ll\cdots\ll\epsilon_p\to 0} \vartheta(\epsilon_1,\dots,\epsilon_p)$$

denote the limit (if it exists and is well-defined) of ϑ along any path $\delta \mapsto \epsilon(\delta)$ towards the origin such that for all $\ell \in \mathbb{N}$ and $j = 2, \dots, p$ there are positive constants $C_{j\ell}$ such that $\epsilon_{j-1}(\delta) \le C_{j\ell}\, \epsilon_j^\ell(\delta)$. Here, we extend the domain of definition of ϑ to points $(0, \dots, 0, \epsilon_{m+1}, \dots, \epsilon_p)$, where $\epsilon_{m+1}, \dots, \epsilon_p > 0$, by defining

$$\vartheta(0, \dots, 0, \epsilon_{m+1}, \dots, \epsilon_p) = \lim_{\epsilon_m \to 0} \dots \lim_{\epsilon_1 \to 0} \vartheta(\epsilon_1, \dots, \epsilon_m, \epsilon_{m+1}, \dots, \epsilon_p).$$

if the limits exist.

Recall that $(\epsilon_1, \dots, \epsilon_p)$ tends to zero along an *admissible paths* in the sense of Coleff-Herrera, [17], if it tends to zero along a path inside $(0, \infty)^p$ such that $\epsilon_{j-1}/\epsilon_j^\ell \to 0$ for all $\ell \in \mathbb{N}$ and $j = 2, \dots, p$. The limits in Definition 1 are (slightly) more general since, e.g., ϵ_1 is allowed to attain the value 0 before the other ϵ_j go to zero. In particular, it thus includes the iterated limit letting $\epsilon_k \to 0$ one at a time. The following theorem is a special case of Theorem 11 below. The proof shares many similarities with the proof of [23, Proposition 1] (even though the statements differ). However, in our case, extra technical difficulties arise since the bundles E_j may have non-trivial metrics.

Theorem 2. *In the sense of currents we have*

$$R^p \wedge \cdots \wedge R^1 = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_p \to 0} R^{p,\epsilon_p} \wedge \cdots \wedge R^{1,\epsilon_1}.$$

To connect with the classical Coleff-Herrera approach, assume temporarily that $\operatorname{rank} E_j = 1$, $j = 1, \dots, p$, so that $R^j = \bar\partial(1/f_j)$. Then Theorem 2 says that for any test form φ of bidegree $(n, n-p)$

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \varphi = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_p \to 0} \int_Z \frac{\bar\partial \chi^{\epsilon_p}}{f_p} \wedge \cdots \wedge \frac{\bar\partial \chi^{\epsilon_1}}{f_1} \wedge \varphi,$$

where $\chi^{\epsilon_j} = \chi(|f_j|^2/\epsilon_j)$. We will refer to the integral on the right hand side as the residue integral and denote is by $\mathcal{I}_f^\varphi(\epsilon)$. If the χ-functions tend to $\mathbf{1}_{[1,\infty)}$ (for a fixed generic $\epsilon \in (0,\infty)^p$) then $\mathcal{I}_f^\varphi(\epsilon)$ tends to Coleff-Herrera's original residue integral

$$I_f^\varphi(\epsilon) = \int_{T(\epsilon)} \varphi/(f_1 \cdots f_p), \tag{1}$$

where $T(\epsilon) = \cap_1^p \{|f_j|^2 = \epsilon_j\}$ is oriented as the distinguished boundary of the corresponding polyhedron. In [17] Coleff and Herrera prove that the limit of $I_f^\varphi(\epsilon)$ along an admissible path exists and defines a current, the nowadays called *Coleff-Herrera product.* We show (see Theorem 11) that the Coleff-Herrera product equals the product $\bar\partial(1/f_p) \wedge \cdots \wedge \bar\partial(1/f_1)$; this is folklore but to our knowledge not completely proved before (except in the case of complete intersection when it follows from [23] and [25] together with [30]).

A result much in the same spirit was proven by Passare in [25], where he relates the original Coleff-Herrera product to residue currents defined by λ-regularizations. Passare considers the regularization

$$\frac{\bar\partial|f_p|^{2\lambda}}{f_p}\wedge\cdots\wedge\frac{\bar\partial|f_1|^{2\lambda}}{f_1}\Big|_{\lambda=0}, \tag{2}$$

i.e., instead of letting the λ_i go to zero successively, all the λ_i are equal to a single λ that tends to 0. In that case, Passare proves that this current coincides with an average of limits along parabolic paths of the residue integral, as considered in [23], irrespectively of whether f defines a complete intersection or not.

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The product $R^k\wedge\cdots\wedge R^1$ does in general not have any natural commutation properties. For instance, $\bar\partial(1/(zw))\wedge\bar\partial(1/z) = 0$ while $\bar\partial(1/z)\wedge\bar\partial(1/(zw)) = \bar\partial(1/z^2)\wedge\bar\partial(1/w)$, where the last product simply is the tensor product. However, if the f_j define a complete intersection, i.e., $\operatorname{codim}\{f_1=\cdots=f_p=0\}=\sum_j \operatorname{rank} E_j$, then it is known (see, e.g., [3]) that the product is commutative; the case when all the E_j have rank 1 is proved in [17].

Remark 3. Recall that the currents R^j take values in ΛE_j. The sum of the degree of R^j in ΛE_j and its form-degree is even. Therefore the product is naturally commutative. If the E_j are trivial line bundles that we do not make any distinction between, then the product is anti-commutative; this is the classical Coleff-Herrera setting.

Theorem 4. *Assume that the f_j define a complete intersection. Then for every test form φ*

$$(\lambda_1,\cdots,\lambda_p)\mapsto\int_Z R^{p,\lambda_p}\wedge\cdots\wedge R^{1,\lambda_1}\wedge\varphi$$

has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^p$.

This result is a special case of our Theorem 14, which generalizes [30, Theorem 1]. The case when $p=2$ and $\operatorname{rank} E_j = 1$ was proved by Berenstein-Yger (see, e.g., [10]). The following result is a special case of Theorem 13, which generalizes [16, Theorem 1].

Theorem 5. *Assume that the f_j define a complete intersection. Then for every test form φ*

$$(\epsilon_1,\ldots,\epsilon_p)\mapsto\int_Z R^{p,\epsilon_p}\wedge\cdots\wedge R^{1,\epsilon_1}\wedge\varphi$$

is Hölder continuous on $[0,\infty)^p$.

For this result it is crucial that the χ-functions used to regularize the R^j are smooth. In fact, Passare-Tsikh, [26], found a quite simple tuple (f_1,f_2) defining a complete intersection in $\mathbb{C}^2$ and a test form

φ such that the classical Coleff-Herrera residue integral $I^{\varphi}_{(f_1,f_2)}(\epsilon)$ is discontinuous at $\epsilon = 0$. Soon after Björk found generic families of such examples, see, e.g., [15].

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Let us give some background and motivation for the kind of products considered here. Products of Cauchy-Fantappié-Leray type currents were first studied by Wulcan, [32]. Wulcan defines the product as the value at $\lambda = 0$ of the analytic continuation of $\lambda \mapsto R^{p,\lambda} \wedge \cdots \wedge R^{1,\lambda}$. In the non-complete intersection case Wulcan's product is different from our; in the case that all E_j have rank 1, $R^{p,\lambda} \wedge \cdots \wedge R^{1,\lambda}|_{\lambda=0}$ coincides with Passare's product, (2). Passare-Wulcan products satisfy several natural computation rules and are quite useful but it has turned out that the recursive definition discussed above often is more natural. In particular, the Stückrad-Vogel intersection algorithm in non-proper intersection theory is conveniently expressed using recursively defined products, see [6].

In the complete intersection case there is no ambiguity, the Coleff-Herrera product is commutative and if $f = (f_1, \dots, f_p)$ then R^f equals $\wedge_j \bar{\partial}(1/f_j)$, see [28] and [1]. This indicates that the Coleff-Herrera product is the "correct" current to associated to a complete intersection. The Coleff-Herrera product is the minimal current extension of Grothendieck's cohomological residue (see, e.g., [24] for definitions) in the sense that it annihilated by anti-holomorphic functions vanishing on its support. Moreover, if f defines a complete intersection then the annihilator ideal of R^f equals the ideal generated by f, see [24] and [18]. This property is very useful and lies behind many applications, e.g., explicit division-interpolation formulas and Briançon-Skoda type results ([2], [10]), explicit versions of the fundamental principle ([13]), the $\bar{\partial}$-equation on complex spaces ([4], [5], [19]), and explicit Green currents in arithmetic intersection theory ([12]).

In Section 2, we give the necessary background and the general formulations of our results. Section 3 contains the proof of Theorems 2 and 11. The proof of Theorems 4, 5, 13 and 14 is the content of Section 4; the crucial part is Lemma 19 which enables us to effectively use the assumption about complete intersection.

2. Formulation of the general results

Let Z be a reduced complex space of pure dimension n. We say that φ is a smooth (p,q)-form on Z if φ is smooth on Z_{reg}, and in a neighborhood of any $p \in Z$, there is a smooth (p,q)-form $\tilde{\varphi}$ in an ambient complex manifold such that the pullback of $\tilde{\varphi}$ to Z_{reg} coincides with $\varphi|_{Z_{reg}}$ close to p. The (p,q)-test forms on Z, $\mathscr{D}_{p,q}(Z)$, are defined as the smooth compactly supported (p,q)-forms (with a suitable topology) and the space of (p,q)-currents on Z, $\mathscr{D}'_{p,q}(Z)$, is the dual of $\mathscr{D}_{n-p,n-q}(Z)$. More concretely, if $i\colon Z \to \Omega \subset \mathbb{C}^N$ is an embedding and μ is a (p,q)-current on Z then $i_*\mu$ is a $(N-n+p, N-n+q)$-current in Ω that vanishes on test forms ξ such that $i^*\xi = 0$ on Z_{reg}. Conversely, such a current in Ω defines a current on Z. See, e.g., [22] for a more thorough discussion.

Let x be a complex coordinate on $\mathbb{C}$. Recall that the principal value current $1/x^m$ can be computed as the value at $\lambda = 0$ of the analytic continuation of $|x|^{2\lambda}/x^m$; the residue current $\bar\partial(1/x^m)$ then is the value at $\lambda = 0$ of $\bar\partial|x|^{2\lambda}/x^m$. Since one can take tensor products of one-variable currents it follows that

$$T = \frac{1}{x_1^{\alpha_1}} \wedge \cdots \wedge \frac{1}{x_p^{\alpha_p}} \wedge \frac{\vartheta(x)}{x_{p+1}^{\alpha_{p+1}} \cdots x_n^{\alpha_n}} \tag{3}$$

is a well defined current in $\mathbb{C}^n$; here $\alpha_1, \dots, \alpha_p$ are positive integers, $\alpha_{p+1}, \dots, \alpha_n$ are non-negative integers, and ϑ is a smooth compactly supported form. Such a current T is called an *elementary pseudomeromorphic* current. Following [7] we say that a current μ on Z is *pseudomeromorphic*, $\mu \in \mathcal{PM}(Z)$, if μ locally is a finite sum of push-forwards $\pi^1_* \cdots \pi^m_* \tau$ under maps

$$X^m \xrightarrow{\pi^m} \cdots \xrightarrow{\pi^2} X^1 \xrightarrow{\pi^1} Z,$$

where each π^j is either a modification or an open inclusion and τ is an elementary pseudomeromorphic current on X^m. It follows that the class of pseudomeromorphic currents is closed under $\bar\partial$ and multiplication with smooth forms, and that the push-forward of a pseudomeromorphic current by a modification is pseudomeromorphic.

Lemma 6. *Let f be a holomorphic function, and let $T \in \mathcal{PM}(Z)$. If $\tilde f$ is a holomorphic function such that $\{\tilde f = 0\} = \{f = 0\}$ and v is a smooth non-zero function, then $(|\tilde f v|^{2\lambda}/f)T$ and $(\bar\partial|\tilde f v|^{2\lambda}/f) \wedge T$ have current-valued analytic continuations to $\lambda = 0$ and the values at $\lambda = 0$ are pseudomeromorphic and independent of the choices of $\tilde f$ and v. Moreover, if $\chi = \mathbf{1}_{[1,\infty)}$, or a smooth approximation thereof, then*

$$\left.\frac{|\tilde f v|^{2\lambda}}{f} T\right|_{\lambda=0} = \lim_{\epsilon\to 0^+} \frac{\chi^\epsilon}{f} T \quad \textit{and} \quad \left.\frac{\bar\partial|\tilde f v|^{2\lambda}}{f} \wedge T\right|_{\lambda=0} = \lim_{\epsilon\to 0^+} \frac{\bar\partial\chi^\epsilon}{f} \wedge T, \tag{4}$$

where $\chi^\epsilon = \chi(|\tilde f v|^2/\epsilon)$.

Proof. The first part is essentially Proposition 2.1 in [7], except that there, Z is a complex manifold, $\tilde f = f$ and $v \equiv 1$. However, with suitable resolutions of singularities, the proof in [7] goes through in the same way in our situation, as long as we observe that in $\mathbb{C}$

$$\frac{|x^{\alpha'} v|^{2\lambda}}{x^\alpha} \frac{1}{x^\beta} \quad \text{and} \quad \frac{|x^{\alpha'} v|^{2\lambda}}{x^\alpha} \bar\partial \frac{1}{x^\beta}$$

have analytic continuations to $\lambda = 0$, and the values at $\lambda = 0$ are $1/x^{\alpha+\beta}$ and 0 respectively, independently of α' and v, as long as $\alpha' > 0$ and $v \neq 0$ (and similarly with $\bar\partial|x^{\alpha'}v|^{2\lambda}/x^{\alpha}$).

By Leibniz rule, it is enough to consider the first equality in (4), since if we have proved the first equality, then

$$\lim_{\epsilon\to 0}\frac{\bar\partial\chi^\epsilon}{f}\wedge T = \lim_{\epsilon\to 0}\bar\partial\left(\frac{\chi^\epsilon}{f}T\right) - \frac{\chi^\epsilon}{f}\bar\partial T$$
$$= \left(\bar\partial\left(\frac{|\tilde f v|^{2\lambda}}{f}T\right) - \frac{|\tilde f v|^{2\lambda}}{f}\bar\partial T\right)\Bigg|_{\lambda=0} = \frac{\bar\partial|\tilde f v|^{2\lambda}}{f}\wedge T\Bigg|_{\lambda=0}.$$

To prove the first equality in (4), we observe first that in the same way as in the first part, we can assume that $f = x^\gamma u$ and $\tilde f = x^{\tilde\gamma}\tilde u$, where u and $\tilde u$ are non-zero holomorphic functions. Since T is a sum of push-forwards of elementary currents, we can assume that T is of the form (3). Note that if $\operatorname{supp}\gamma\cap\operatorname{supp}\beta\neq\emptyset$, then $(|\tilde f v|^{2\lambda}/f)T = 0$ for $\mathfrak{Re}\,\lambda \gg 0$ and $(\chi(|\tilde f v|^2/\epsilon)/f)T = 0$ for $\epsilon > 0$, since $\operatorname{supp} T \subseteq \{x_i = 0, i \in \operatorname{supp}\beta\}$. Thus, we can assume that $\operatorname{supp}\gamma\cap\operatorname{supp}\beta = \emptyset$. By a smooth (but non-holomorphic) change of variables, as in Section 3 (equations (13)), we can assume that $|\tilde u v|^2 \equiv 1$. Thus, since $(|x^{\tilde\gamma}|^{2\lambda}/x^\gamma)(1/x^\alpha)$, $(\chi(|x^{\tilde\gamma}|^2/\epsilon)/x^\gamma)(1/x^\alpha)$ depend on variables disjoint from the ones that $\wedge_{\beta_i\neq 0}\bar\partial(1/x_i^{\beta_i})$ depends on, it is enough to prove that

$$\frac{|x^{\tilde\gamma}|^{2\lambda}}{x^\gamma}\frac{1}{x^\alpha}\Bigg|_{\lambda=0} = \lim_{\epsilon\to 0}\frac{\chi(|x^{\tilde\gamma}|^2/\epsilon)}{x^\gamma}\frac{1}{x^\alpha},$$

which is Lemma 2 in [16]. □

Let $E_1,\dots,E_q$ be holomorphic hermitian vector bundles over Z, let f_j be a holomorphic section of E_j^*, $j = 1,\dots,q$, and let s_j be the section of E_j with pointwise minimal norm such that $f_j\cdot s_j = |f_j|^2$. Outside $\{f_j = 0\}$, define

$$u_k^j = \frac{s_j\wedge(\bar\partial s_j)^{k-1}}{|f_j|^{2k}}.$$

It is easily seen that if $f_j = f_j^0 f_j'$, where f_j^0 is a holomorphic function and f_j' is a non-vanishing section, then $u_k^j = (1/f_j^0)^k(u')_k^j$, where $(u')_k^j$ is smooth across $\{f_j = 0\}$. We let

$$U^j = \sum_{k=1}^{\infty}|\tilde f_j|^{2\lambda}u_k^j\Big|_{\lambda=0}, \tag{5}$$

where $\tilde f_j$ is any holomorphic section of E_j^* such that $\{\tilde f_j = 0\} = \{f_j = 0\}$. The existence of the analytic continuation is a local statement, so we can assume that $f_j = \sum f_{j,k}\mathfrak{e}_{j,k}^*$, where $\mathfrak{e}_{j,k}^*$ is a local holomorphic frame for E_j^*. After principalization we can assume that the ideal $\langle f_{j,1},\dots,f_{j,k_j}\rangle$ is generated by, e.g., $f_{j,0}$. By the representation $u_k^j =$

$(1/f_{j,0})^k(u')^j_k$, the existence of the analytic continuation of U^j in (5) then follows from Lemma 6. Let U^j_k denote the term of U^j that takes values in $\Lambda^k E_j$; U^j_k is thus a $(0,k-1)$-current with values in $\Lambda^k E_j$. Let δ_{f_j} denote interior multiplication with f_j and put $\nabla_{f_j} = \delta_{f_j} - \bar\partial$; it is not hard to verify that $\nabla_{f_j} U = 1$ outside $f_j = 0$. We define the Cauchy-Fantappiè-Leray type residue current, R^j, of f_j by $R^j = 1 - \nabla_{f_j} U^j$. One readily checks that

$$
\begin{aligned}
(6) \qquad R^j &= R^j_0 + \sum_{k=1}^{\infty} R^j_k \\
&= (1 - |\tilde f_j|^{2\lambda})|_{\lambda=0} + \sum_{k=1}^{\infty} \bar\partial |\tilde f_j|^{2\lambda} \wedge \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}}\bigg|_{\lambda=0},
\end{aligned}
$$

where, as above, $\tilde f_j$ is a holomorphic section such that $\{\tilde f_j = 0\} = \{f_j = 0\}$.

Remark 7. Notice that if E_j has rank 1, then U_j simply equals $1/f_j$ and $R^j = 1 - \nabla_{f_j}(1/f_j) = 1 - f_j \cdot (1/f_j) + \bar\partial(1/f_j) = \bar\partial(1/f_j)$.

We now define a non-commutative calculus for the currents U^i_k and R^j_ℓ recursively as follows.

Definition 8. If T is a product of some U^i_k and R^j_ℓ, then we define

- $U^j_k \wedge T = |\tilde f_j|^{2\lambda} \dfrac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}} \wedge T\bigg|_{\lambda=0}$
- $R^j_0 \wedge T = (1 - |\tilde f_j|^{2\lambda}) T\big|_{\lambda=0}$
- $R^j_k \wedge T = \bar\partial |\tilde f_j|^{2\lambda} \wedge \dfrac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}} \wedge T\bigg|_{\lambda=0}$,

where $\tilde f_j$ is any holomorphic section of E^*_j with $\{\tilde f_j = 0\} = \{f_j = 0\}$.

Notice that after principalization the pull-back of u^j_k is semi-meromorphic; in particular U^j and R^j are pseudomeromorphic. Thus, by Lemma 6, the analytic continuations of Definition 8 exist and the values at $\lambda = 0$ are pseudomeromorphic as well.

Remark 9. Under assumptions about complete intersection, these products have the suggestive commutation properties, e.g., if $\operatorname{codim} \{f_i = f_j = 0\} = \operatorname{rank} E_i + \operatorname{rank} E_j$, then $R^i_k \wedge R^j_\ell = R^j_\ell \wedge R^i_k$, $R^i_k \wedge U^j_\ell = U^j_\ell \wedge R^i_k$, and $U^i_k \wedge U^j_\ell = -U^j_\ell \wedge U^i_k$, (see, e.g., [3]). In general, there are no simple relations. However, products involving only U:s are always anti-commutative.

Now, consider collections $R = \{R^1_{k_1}, \dots, R^p_{k_p}\}$ and $U = \{U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q}\}$ and put $(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$. For a permutation ν of $\{1, \dots, q\}$ we define

$$(UR)^\nu = P_{\nu(q)} \wedge \cdots \wedge P_{\nu(1)}. \tag{7}$$

From (5) and (6) we get natural λ-regularizations, P^λ_j, of P_j and from Definition 8 we have $(UR)^\nu = P^{\lambda_q}_{\nu(q)} \wedge \cdots \wedge P^{\lambda_1}_{\nu(1)}|_{\lambda_1=0} \cdots |_{\lambda_q=0}$, i.e., we set successively $\lambda_1 = 0$, then $\lambda_2 = 0$ and so on. The following result is proved in [6].

Theorem 10. *Let $a_1 > \cdots > a_q > 0$ be integers and λ a complex variable. Then*

$$\lambda \mapsto P^{\lambda^{a_q}}_{\nu(q)} \wedge \cdots \wedge P^{\lambda^{a_1}}_{\nu(1)}$$

has a current-valued analytic continuation to a neighborhood of the half-axis $[0, \infty) \subset \mathbb{C}$ and the value at $\lambda = 0$ equals $(UR)^\nu$.

The recursively defined product $(UR)^\nu$ can thus be obtained as the value at zero of a one-variable ζ-type function. From an algebraic point of view, this is desirable since one can derive functional equations and use Bernstein-Sato theory to study $(UR)^\nu$.

There are also more concrete and explicit regularizations of the currents U^i_k and R^j_ℓ inspired by [17] and [23]. Let $\chi = \mathbf{1}_{[1,\infty)}$, or a smooth approximation thereof that is 0 close to 0 and 1 close to ∞. It follows from [29], or after principalization from Lemma 6, that

$$U^j_k = \lim_{\epsilon \to 0^+} \chi(|\tilde{f}_j|^2/\epsilon) \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}}. \tag{8}$$

$$R^j_k = \lim_{\epsilon \to 0^+} \bar\partial \chi(|\tilde{f}_j|^2/\epsilon) \wedge \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}}, \; k > 0, \tag{9}$$

and similarly for $k = 0$; as usual, $\{\tilde{f}_j = 0\} = \{f_j = 0\}$. Of course, the limits are in the current sense and if $\chi = \mathbf{1}_{[1,\infty)}$, then ϵ is supposed to be a regular value for $|f_j|^2$ and $\bar\partial \chi(|f_j|^2/\epsilon)$ is to be interpreted as integration over the manifold $|f_j|^2 = \epsilon$. We denote the regularizations given by (8) and (9) by P^ϵ_j.

Theorem 11. *Let $R = \{R^1_{k_1}, \dots, R^p_{k_p}\}$ and $U = \{U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q}\}$ be collections of currents defined in (5) and (6). Let ν be a permutation of $\{1, \dots, q\}$ and let $(UR)^\nu$ be the product defined in (7). Then*

$$(UR)^\nu = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} P^{\epsilon_q}_{\nu(q)} \wedge \cdots \wedge P^{\epsilon_1}_{\nu(1)},$$

where, as above, $(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$; see Definition 1 for the meaning of the limit. If $\chi = \mathbf{1}_{[1,\infty)}$, we require that $\epsilon \to 0$ along an admissible path in the sense of Coleff-Herrera.

Thus $(UR)^\nu$ can be computed as the weak limit of an explicit smooth form and moreover, Definition 8 give the Coleff-Herrera product (in case the bundles E_j have rank 1).

Remark 12. It might be more natural to consider products of whole Cauchy-Fantappiè-Leray type currents, U^j and R^j, as in (5) and (6), and not just products of their components U^j_k and R^j_k, cf., for example [6]. However, since such a product is a sum of products of their components, it follows readily that Theorem 11 holds also for products of whole Cauchy-Fantappiè-Leray type currents.

2.1. **The complete intersection case.** Assume that $f_1, \dots, f_q$ define a complete intersection, i.e., that $\operatorname{codim}\{f_1 = \cdots = f_q = 0\} = \operatorname{rank} E_1 + \cdots + \operatorname{rank} E_q$. Then we know that the calculus defined in Definition 8 satisfies the suggestive commutation properties, but we have in fact the following much stronger results.

Theorem 13. *Assume that $f_1, \dots, f_q$ defines a complete intersection on Z, let $(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$, and let $P_j^{\epsilon_j}$ be an ϵ-regularization of P_j defined by (8) and (9) with smooth χ-functions. Then we have*

$$\left| \int_Z P_1^{\epsilon_1} \wedge \cdots \wedge P_q^{\epsilon_q} \wedge \varphi - P_1 \wedge \cdots \wedge P_q . \varphi \right| \le C \|\varphi\|_{C^M} (\epsilon_1^\omega + \cdots + \epsilon_q^\omega),$$

where M and ω only depend on $f_1, \dots, f_q$, Z, and $\operatorname{supp}\varphi$ while C also depends on the C^M-norm of the χ-functions.

Theorem 14. *Assume that $f_1, \dots, f_q$ defines a complete intersection on Z, let $(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$, and let $P_j^{\lambda_j}$ be the λ-regularization of P_j given by (5) and (6). Then the current valued function*

$$\lambda \mapsto P_1^{\lambda_1} \wedge \cdots \wedge P_q^{\lambda_q},$$

a priori defined for $\mathfrak{Re}\, \lambda_j \gg 0$, has an analytic continuation to a neighborhood of the half-space $\cap_1^q \{\mathfrak{Re}\, \lambda_j \ge 0\}$.

Remark 15. In case the E_j are trivial with trivial metrics, Theorems 13 and 14 follow quite easily from, respectively, [16, Theorem 1] and [30, Theorem 1] by taking averages. As an illustration, let $\varepsilon_1, \dots, \varepsilon_r$ be a nonsense basis and let $f_1, \dots, f_r$ be holomorphic functions. Then we can write $s = \bar{f} \cdot \varepsilon$ and so $u_k = (\bar{f} \cdot \varepsilon) \wedge (d\bar{f} \cdot \varepsilon)^{k-1} / |f|^{2k}$. A standard computation shows that

$$\int_{\alpha \in \mathbb{CP}^{r-1}} \frac{|\alpha \cdot f|^{2\lambda} \alpha \cdot \varepsilon}{(\alpha \cdot f)|\alpha|^{2\lambda}} dV = A(\lambda) |f|^{2\lambda} \frac{\bar{f} \cdot \varepsilon}{|f|^2},$$

where dV is the (normalized) Fubini-Study volume form and A is holomorphic with $A(0) = 1$. It follows that

$$\int_{\alpha_1,\ldots,\alpha_k\in\mathbb{CP}^{r-1}} \bigwedge_1^k \frac{\bar\partial|\alpha_j\cdot f|^{2\lambda}}{\alpha_j\cdot f}\wedge\frac{\alpha_j\cdot\varepsilon}{|\alpha_j|^{2\lambda}}dV(\alpha_j) = A(\lambda)^k\bar\partial(|f|^{2k\lambda}u_k).$$

Elaborating this formula and using [30, Theorem 1] one can show Theorem 14 in the case of trivial E_j with trivial metrics. The general case can probably also be handled in a similar manner but the computations become more involved and we prefer to give direct proofs.

3. Proof of Theorem 11

We start by making a Hironaka resolution of singularities, [21], of Z such that the pre-image of $\cup_j\{f_j = 0\}$ has normal crossings. We then make further toric resolutions (e.g., as in [28]) such that, in local charts, the pullback of each f_i is a monomial, x^{α_i}, times a non-vanishing holomorphic tuple. One checks that the pullback of P_j^ϵ is of one of the following forms:

$$\frac{\chi(|x^{\tilde\alpha}|^2\xi/\epsilon)}{x^\alpha}\vartheta,\quad 1-\chi(|x^{\tilde\alpha}|^2\xi/\epsilon),\quad \frac{\bar\partial\chi(|x^{\tilde\alpha}|^2\xi/\epsilon)}{x^\alpha}\wedge\vartheta,$$

where ξ is smooth and positive, $\operatorname{supp}\tilde\alpha = \operatorname{supp}\alpha$, and ϑ is a smooth bundle valued form; by localizing on the blow-up we may also suppose that ϑ has as small support as we wish. If the χ-functions are smooth, the following special case of Theorem 11 now immediately follows from Lemma 6:

$$(UR)^\nu = \lim_{\epsilon_q\to 0}\cdots\lim_{\epsilon_1\to 0} P_{\nu(q)}^{\epsilon_q}\wedge\cdots\wedge P_{\nu(1)}^{\epsilon_1}. \tag{10}$$

For smooth χ-functions we put

$$\mathcal{I}(\epsilon) = \int\frac{\bar\partial\chi_1^\epsilon\wedge\cdots\wedge\bar\partial\chi_p^\epsilon\chi_{p+1}^\epsilon\cdots\chi_q^\epsilon}{x^{\alpha_1+\cdots+\alpha_p+\cdots+\alpha_{q'}}}\wedge\varphi,$$

where $q'\le q$, φ is a smooth $(n, n-p)$-form with support close to the origin, and $\chi_j^\epsilon = \chi(|x^{\tilde\alpha_j}|^2\xi_j/\epsilon_j)$ for smooth positive ξ_j. We note that we may replace the $\bar\partial$ in $\mathcal{I}(\epsilon)$ by d for bidegree reasons. In case $\chi = \mathbf{1}_{[1,\infty)}$ we denote the corresponding integral by $I(\epsilon)$. We also put $\mathcal{I}^\nu(\epsilon_1,\ldots,\epsilon_q) = \mathcal{I}(\epsilon_{\nu(1)},\ldots,\epsilon_{\nu(q)})$ and similarly for I^ν. In view of (10), the special case of Theorem 11 when the χ-functions are smooth will be proved if we can show that

$$\lim_{\epsilon_1\ll\cdots\ll\epsilon_q\to 0}\mathcal{I}^\nu(\epsilon) \tag{11}$$

exists. The case with $\chi = \mathbf{1}_{[1,\infty)}$ will then follow if we can show

$$\lim_{\delta\to 0}(\mathcal{I}^\nu(\epsilon(\delta)) - I^\nu(\epsilon(\delta))) = 0, \tag{12}$$

where $\delta \mapsto \epsilon(\delta)$ is any admissible path.

For notational convenience, we will consider $\mathcal{I}^\nu(\epsilon)$ (unless otherwise stated), but our arguments apply just as well to $I^\nu(\epsilon)$ until we arrive at the integral (16).

Denote by $\tilde{A}$ the $q \times n$-matrix with rows $\tilde{\alpha}_i$. We will first show that we can assume that $\tilde{A}$ has full rank. The idea is the same as in [17] and [23], however because of the paths along which our limits are taken, we have to modify the argument slightly. The following lemma follows from the proof of Lemma III.12.1 in [31].

Lemma 16. *Assume that α is a $q \times n$-matrix with rows α_i such that there exists $(v_1, \dots, v_q) \neq 0$ with $\sum v_i \alpha_i = 0$. Let $j = \min\{i; v_i \neq 0\}$. Then there exist constants $C, c > 0$ such that if $\epsilon_j < C(\epsilon_{j+1} \dots \epsilon_q)^c$, then $\chi(|x^{\alpha_j}|^2 \xi_j/\epsilon_j) \equiv 1$ and $\bar{\partial}\chi(|x^{\alpha_j}|^2 \xi_j/\epsilon_j) \equiv 0$ for all $x \in \Delta \cap \{|x^{\alpha_i}|^2 \geq C_i \epsilon_i, i = j+1, \dots, q\}$, where Δ is the unit polydisc.*

Assume that $\tilde{A}$ does not have full rank, and let v be a column vector such that $v^t \tilde{A} = 0$. Since $(\epsilon_1, \dots, \epsilon_q)$ is replaced by $(\epsilon_{\nu(1)}, \dots, \epsilon_{\nu(q)})$ in $\mathcal{I}^\nu(\epsilon)$, we choose instead j_0 such that $\nu(j_0) \leq \nu(i)$ for all i such that $v_i \neq 0$. If $j_0 \leq p$, we let $\widetilde{\mathcal{I}}^\nu(\epsilon) = 0$, and if $j_0 \geq p+1$, we let $\widetilde{\mathcal{I}}^\nu(\epsilon)$ be $\mathcal{I}^\nu(\epsilon)$ but with $\chi^\epsilon_{j_0}$ replaced by 1. If $\epsilon = \epsilon(\delta)$ is such that $\epsilon_{\nu(j_0)} > 0$, then $\mathcal{I}^\nu(\epsilon)$ is a current acting on a test form with support on a set of the form

$$\Delta \cap \{|x^{\alpha_i}|^2 \geq C_i \epsilon_{\nu(i)}; \text{for all } i \text{ such that } \nu(i) \geq \nu(j_0)\}.$$

In particular, if $\epsilon_{\nu(j_0)}(\delta)$ is sufficiently small compared to $(\epsilon_{\nu(j_0)+1}(\delta), \dots, \epsilon_q(\delta))$, then by Lemma 16, if $j_0 \leq p$, the factor $\bar{\partial}\chi^\epsilon_{j_0}$ is identically 0, and if $j_0 \geq p+1$, the factor $\chi^\epsilon_{j_0}$ is identically 1 and thus is equal to $\widetilde{\mathcal{I}}^\nu(\epsilon)$ for such ϵ. Similarly, if $\epsilon_{\nu(j_0)} = 0$, we have that $\mathcal{I}^\nu(\epsilon)$ is defined as a limit along $\epsilon_{\nu(j_0)} \to 0$, with $\epsilon_{\nu(j_0)+1}, \dots, \epsilon_q$ fixed and in the limit we get again that for sufficiently small $\epsilon_{\nu(j_0)}$, we can replace $\mathcal{I}^\nu(\epsilon)$ by $\widetilde{\mathcal{I}}^\nu(\epsilon)$. Thus we have

$$\lim_{\epsilon_1 \ll \dots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \lim_{\epsilon_1 \ll \dots \ll \epsilon_q \to 0} \widetilde{\mathcal{I}}^\nu(\epsilon),$$

and we have reduced to the case that $\tilde{A}$ is a $(q-1) \times n$-matrix of the same rank. We continue this procedure until $\tilde{A}$ has full rank.

By re-numbering the coordinates, we may suppose that the minor $A = (\tilde{\alpha}_{ij})_{1 \leq i,j \leq q}$ of $\tilde{A}$ is invertible and we put $A^{-1} = B = (b_{ij})$. We now use complex notation to make a non-holomorphic, but smooth change of variables:

$$y_1 = x_1\, \xi^{b_1/2}, \dots, y_q = x_q\, \xi^{b_q/2}, y_{q+1} = x_{q+1}, \dots, y_n = x_n, \tag{13}$$

$$\bar{y}_1 = \bar{x}_1\, \xi^{b_1/2}, \dots, \bar{y}_q = \bar{x}_q\, \xi^{b_q/2}, \bar{y}_{q+1} = \bar{x}_{q+1}, \dots, \bar{y}_n = \bar{x}_n,$$

where $\xi^{b_i/2} = \xi_1^{b_{i1}/2} \cdots \xi_q^{b_{iq}/2}$. One easily checks that $dy \wedge d\bar{y} = \xi^{b_1} \cdots \xi^{b_q}$ $dx \wedge d\bar{x} + O(|x|)$, so (13) defines a smooth change of variables between neighborhoods of the origin. A simple linear algebra computation then shows that $|x^{\tilde{\alpha}_i}|^2 \xi_i = |y^{\tilde{\alpha}_i}|^2$. Of course, this change of variables does not preserve bidegrees so $\varphi(y)$ is merely a smooth compactly supported $(2n-p)$-form. We thus have

$$\mathcal{I}^{\nu}(\epsilon) = \int_{\Delta} \frac{d\chi_1^{\epsilon} \wedge \cdots \wedge d\chi_p^{\epsilon} \chi_{p+1}^{\epsilon} \cdots \chi_q^{\epsilon}}{y^{\alpha_1 + \cdots + \alpha_p + \cdots + \alpha_{q'}}} \wedge \varphi'(y), \tag{14}$$

where $\chi_j^{\epsilon} = \chi(|y^{\tilde{\alpha}_j}|^2 / \epsilon_{\nu(j)})$ and $\varphi'(y) = \sum_{|I|+|J|=2n-p} \psi_{IJ} dy_I \wedge d\bar{y}_J$. By linearity we may assume that the sum only consists of one term $\varphi'(y) = \psi dy_K \wedge d\bar{y}_L$, and by scaling, we may assume that $\operatorname{supp} \psi \subseteq \Delta$, Δ being the unit polydisc. By Lemma 2.4 in [17], we can write the function ψ as

$$\psi(y) = \sum_{I+J < \sum_1^{q'} \alpha_j - \mathbf{1}} \psi_{IJ}\, y^I \bar{y}^J + \sum_{I+J = \sum_1^{q'} \alpha_j - \mathbf{1}} \psi_{IJ}\, y^I \bar{y}^J, \tag{15}$$

where $a < b$ for tuples a and b means that $a_i < b_i$ for all i. In the decomposition (15) each of the smooth functions ψ_{IJ} in the first sum on the left-hand side is independent of some variable. We now show that this implies that the first sum on the left-hand side of (15) does not contribute to the integral (14). In case $\varphi'(y)$ has bidegree $(n, n-p)$ this is a well-known fact but we must show it for an arbitrary $(2n-p)$-form.

We change to polar coordinates:

$$dy_K \wedge d\bar{y}_L = d(r_{K_1} e^{i\theta_{K_1}}) \wedge \cdots \wedge d(r_{L_1} e^{-i\theta_{L_1}}) \wedge \cdots$$

Since χ_j^{ϵ} in (14) is independent of θ, it follows that we must have full degree $= n$ in $d\theta$. The only terms in the expansion of $dy_K \wedge d\bar{y}_L$ above that will contribute to (14) are therefore of the form

$$c\, r_1 \cdots r_n e^{i\theta\cdot\gamma}\, dr_M \wedge d\theta,$$

where $|M| = n - p$, c is a constant, and γ is a multiindex with entries equal to 1, -1, or 0. Substituting this and a term $\psi_{IJ} y^I \bar{y}^J = \psi_{IJ} r^{I+J} e^{i\theta\cdot(I-J)}$ from (15) into (14) gives rise to an "inner" θ-integral (by Fubini's theorem):

$$\mathscr{I}_{IJ}(r) = \int_{\theta \in [0,2\pi)^n} \psi_{IJ}(r,\theta)\; e^{i\theta\cdot(I-J-\sum_1^{q'} \alpha_j + \gamma)}\, d\theta.$$

If $I+J < \sum_1^{q'} \alpha_j - \mathbf{1}$, then $I-J-\sum_1^{q'} \alpha_j + \gamma < 0$ and ψ_{IJ} is independent of some $y_j = r_j e^{i\theta_j}$. Integrating over $\theta_j \in [0, 2\pi)$ thus yields $\mathscr{I}_{IJ} = 0$ if $I + J < \sum_1^{q'} \alpha_j - \mathbf{1}$. If instead $I + J = \sum_1^{q'} \alpha_j - \mathbf{1}$, then $\mathscr{I}_{IJ}(r)$ is smooth on $[0,\infty)^n$.

Summing up, we see that we can write (14) as

$$\mathcal{I}^\nu(\epsilon) = \int_{r\in(0,1)^n} d\chi_1^\epsilon \wedge \cdots \wedge d\chi_p^\epsilon \chi_{p+1}^\epsilon \cdots \chi_q^\epsilon \mathscr{J}(r)\, dr_M, \tag{16}$$

where $\chi_j^\epsilon = \chi(r^{2\alpha_j}/\epsilon_{\nu(j)})$, $\mathscr{J}$ is smooth, and $|M| = n-p$.

After these reductions, the integral (16) we arrive at is the same as equation (16) in [23], and we will use the fact proven there, that $\lim_{\delta\to 0} \mathcal{I}^\nu(\epsilon(\delta))$ exists along any admissible path $\epsilon(\delta)$, and is well-defined independently of the choice of admissible path. (This is not exactly what is proven there, but the fact that if $b \in \mathbb{Q}^p$, then $\lim_{\delta\to 0}\epsilon(\delta)^b$ is either 0 or ∞ independently of the admissible path chosen is the only addition we need to make for the argument to go through in our case.) Using this, if we let $\epsilon(\delta)$ be any admissible path, we will show by induction over q that

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \lim_{\delta\to 0} \mathcal{I}^\nu(\epsilon(\delta)).$$

For $q = 1$ this is trivially true, so we assume $q > 1$. Let ϵ^k be any sequence satisfying the conditions in Definition 1. Consider a fixed k, and let m be such that $\epsilon^k = (0,\dots,0,\epsilon^k_{m+1},\dots,\epsilon^k_q)$ with $\epsilon^k_{m+1} > 0$. Let $I_1 = \nu^{-1}(\{1,\dots,m\}) \cap \{1,\dots,p\}$ and $I_2 = \nu^{-1}(\{1,\dots,m\}) \cap \{p+1,\dots,q\}$. We consider $\epsilon^k_{m+1},\dots,\epsilon^k_q$ fixed in $\mathcal{I}^\nu(\epsilon)$, and define

$$\mathcal{I}_k(\epsilon_1,\dots,\epsilon_m) = \int_{[0,1]^n} \bigwedge_{i\in I_1} d\chi(r^{\alpha_i}/\epsilon_{\nu(i)}) \prod_{i\in I_2} \chi(r^{\alpha_i}/\epsilon_{\nu(i)}) \mathscr{J}_k(r) dr_M,$$

originally defined on $(0,\infty)^p$, but extended according to Definition 1, where

$$\mathscr{J}_k(r) = \pm \bigwedge_{i\in\{1,\dots,p\}\setminus I_1} d\chi(r^{\alpha_i}/\epsilon^k_{\nu(i)}) \prod_{i\in\{p+1,\dots,q\}\setminus I_2} \chi(r^{\alpha_i}/\epsilon^k_{\nu(i)}) \mathscr{J}(r)$$

(where the sign is chosen such that $\mathcal{I}_k(0) = \mathcal{I}^\nu(\epsilon^k)$). Since $m < q$ and $\mathscr{J}_k$ is smooth, we have by induction that

$$\mathcal{I}_k(0) = \lim_{\epsilon_m\to 0} \dots \lim_{\epsilon_1\to 0} \mathcal{I}_k(\epsilon_1,\dots,\epsilon_m) = \lim_{\delta\to 0} \mathcal{I}_k(\epsilon'(\delta)),$$

where $\epsilon'(\delta)$ is any admissible path, and the first equality follows by definition of $\mathcal{I}_k(0)$. We fix an admissible path $\epsilon'(\delta)$. For each k we can choose δ_k such that if $\epsilon^{k'} = (\epsilon'_1(\delta_k),\dots,\epsilon'_m(\delta_k))$, then $\lim_{k\to\infty}(\mathcal{I}_k(\epsilon^{k'}) - \mathcal{I}_k(0)) = 0$ and if $\tilde\epsilon^k = (\epsilon^{k'},\epsilon^k_{m+1},\dots,\epsilon^k_q)$, then $\tilde\epsilon^k$ forms a subsequence of an admissible path. Since $\mathcal{I}_k(0) = \mathcal{I}^\nu(\epsilon^k)$, and $\mathcal{I}_k(\epsilon^{k'}) = \mathcal{I}^\nu(\tilde\epsilon^k)$, we thus have

$$\lim_{k\to\infty} \mathcal{I}^\nu(\epsilon^k) = \lim_{k\to\infty} \mathcal{I}^\nu(\tilde\epsilon^k) = \lim_{\delta\to 0} \mathcal{I}^\nu(\epsilon(\delta))$$

where the second equality follows from the existence and uniqueness of $\mathcal{I}^\nu(\epsilon(\delta))$ along any admissible path. Hence we have shown that the limit in (11) exists and is well-defined.

Finally, if we start from (16), as (23) in [23] shows, either

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \pm \int_{r_M \in (0,1)^{n-p}} \mathscr{J}(0, r_M) dr_M,$$

or the limit is 0, depending only on α. If we consider $I^\nu(\epsilon)$ instead, we get the same limit, see [31, p. 79–80], and (12) follows.

4. Proof of Theorems 13 and 14

As in [30] and [16] the key-step of the proof is a Whitney type division lemma, Lemma 19 below. Recall that

$$(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$$

and that $P_j^{\epsilon_j}$ and $P_j^{\lambda_j}$ are the ϵ-regularizations with smooth χ (given by (8), (9)) and the λ-regularizations (cf., (5), (6)) respectively of P_j. We will consider the following two integrals:

$$\mathcal{I}(\epsilon) = \int_Z P_1^{\epsilon_1} \wedge \cdots \wedge P_q^{\epsilon_q} \wedge \varphi$$

$$\Gamma(\lambda) = \int_Z P_1^{\lambda_1} \wedge \cdots \wedge P_q^{\lambda_q} \wedge \varphi,$$

where φ is a test form on Z, supported close to a point in $\{f_1 = \cdots = f_q = 0\}$, of bidegree $(n, n - k_1 - \cdots - k_q + q - p)$ with values in $\Lambda(E_1^* \oplus \cdots \oplus E_q^*)$. In the arguments below, we will assume for notational convenience that $\tilde{f}_j = f_j$ (cf., e.g., (5)); the modifications to the general case are straightforward.

The main parts of the proofs of Theorems 13 and 14 are contained in the following propositions.

Proposition 17. *Assume that $f_1, \dots, f_q$ define a complete intersection. For $p < s \leq q$ we have*

$$\left|\mathcal{I}(\epsilon) - \mathcal{I}(\epsilon_1, \dots, \epsilon_{s-1}, 0, \dots, 0)\right| \leq C\|\varphi\|_M(\epsilon_s^\omega + \cdots + \epsilon_q^\omega).$$

Note that $\mathcal{I}(\epsilon_1, \dots, \epsilon_{s-1}, 0, \dots, 0)$ is well-defined; it is the action of $U^s_{k_s} \wedge \cdots \wedge U^q_{k_q}$ on a smooth form.

Proposition 18. *Assume that $f_1, \dots, f_q$ define a complete intersection. Then $\Gamma(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^q$ and its only possible poles in a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \geq 0\}$ are along hyperplanes of the form $\sum_{j=1}^p \lambda_j \alpha_j = 0$, where $\alpha_j \in \mathbb{N}$ and at least two α_j are positive. In particular, for $p = 1$, $\Gamma(\lambda)$ is analytic in a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \geq 0\}$.*

Using that

$$\bar\partial |f_j|^{2\lambda} \wedge u_k^j = \bar\partial(|f_j|^{2\lambda} u_k^j) - f_j \cdot (|f_j|^{2\lambda} u_{k+1}^j), \tag{17}$$

the proof of Theorem 14 follows from Proposition 18 in a similar way as Theorem 1 in [30] follows from Proposition 4 in [30].

We indicate one way Proposition 17 can be used to prove Theorem 13. To simplify notation somewhat, we let R^j denote any R^j_k and R^j_ϵ denotes a smooth ϵ-regularization of R^j; U^j and U^j_ϵ are defined similarly. The uniformity in the estimate of Proposition 17 implies that we have estimates of the form

$$\left|\bigwedge_1^m R^j_\epsilon \wedge \bigwedge_{m+1}^p R^j \wedge \bigwedge_{p+1}^q U^j_\epsilon - \bigwedge_1^m R^j_\epsilon \wedge \bigwedge_{m+1}^p R^j \wedge \bigwedge_{p+1}^q U^j\right| \lesssim (\epsilon_{p+1}^\omega + \cdots + \epsilon_q^\omega), \tag{18}$$

where, e.g., $R^{m+1}\wedge\cdots\wedge R^p$ a priori is defined as a Coleff-Herrera product. We prove (a slightly stronger result than) Theorem 13 by induction over p. Let R^* denote the Coleff-Herrera product of some R^j:s with $j>p$ and let U^* and U^*_ϵ denote the product of some U^j:s and U^j_ϵ:s respectively, also with $j>p$ but only j:s not occurring in R^*. We prove

$$\left|R^1_\epsilon\wedge\cdots\wedge R^p_\epsilon\wedge R^*\wedge U^*_\epsilon - R^1\wedge\cdots\wedge R^p\wedge R^*\wedge U^*\right| \lesssim \epsilon^\omega,$$

i.e., we prove Theorem 13 *on* the current R^*. The induction start, $p=0$, follows immediately from (18). If we add and subtract $R^1_\epsilon\wedge\cdots\wedge R^p_\epsilon\wedge R^*\wedge U^*$, the induction step follows easily from (17) (construed in setting of ϵ-regularizations) and estimates like (18).

Proof of Propositions 17 and 18. We may assume that φ has arbitrarily small support. Hence, we may assume that Z is an analytic subset of a domain $\Omega\subseteq\mathbb{C}^N$ and that all bundles are trivial, and thus make the identification $f_j=(f_{j1},\dots,f_{je_j})$, where f_{ji} are holomorphic in Ω. We choose a Hironaka resolution $\hat{Z}\to Z$ such that the pulled-back ideals $\langle\hat{f}_j\rangle$ are all principal, and moreover, so that in a fixed chart with coordinates x on $\hat{Z}$ (and after a possible re-numbering), $\langle\hat{f}_j\rangle$ is generated by $\hat{f}_{j1}$ and $\hat{f}_{j1}=x^{\alpha_j}h_j$, where h_j is holomorphic and non-zero. We then have

$$|\hat{f}_j|^2=|\hat{f}_{j1}|^2\xi_j,\quad \hat{u}^j_{k_j}=v^j/\hat{f}_{j1}^{k_j},$$

where ξ_j is smooth and positive and v^j is a smooth (bundle valued) form. We thus get

$$\bar\partial\chi_j(|\hat{f}_j|^2/\epsilon_j)=\tilde{\chi}_j(|\hat{f}_j|^2/\epsilon_j)\left(\frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}}+\frac{\bar\partial\xi_j}{\xi_j}\right),$$

where $\tilde{\chi}_j(t)=t\chi_j'(t)$, and

$$\bar\partial|\hat{f}_j|^{2\lambda_j}=\lambda_j|\hat{f}_j|^{2\lambda_j}\left(\frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}}+\frac{\bar\partial\xi_j}{\xi_j}\right).$$

It follows that $\mathcal{I}(\epsilon)$ and $\Gamma(\lambda)$ are finite sums of integrals which we without loss of generality can assume to be of the form

$$\pm\int_{\mathbb{C}^n_x}\prod_1^p\tilde{\chi}_j^\epsilon\prod_{p+1}^q\chi_j^\epsilon\bigwedge_1^m\frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}}\wedge\bigwedge_{m+1}^p\frac{\bar\partial\xi_j}{\xi_j}\wedge\bigwedge_1^q\frac{v^j}{\hat{f}_{j1}^{k_j}}\wedge\varphi\rho, \tag{19}$$

$$\pm\lambda_1\cdots\lambda_p\int_{\mathbb{C}^n_x}\prod_1^q|\hat{f}_j|^{2\lambda_j}\bigwedge_1^m\frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}}\wedge\bigwedge_{m+1}^p\frac{\bar\partial\xi_j}{\xi_j}\wedge\bigwedge_1^q\frac{v^j}{\hat{f}_{j1}^{k_j}}\wedge\varphi\rho, \tag{20}$$

where ρ is a cutoff function.

Recall that $\hat{f}_{j1}=x^{\alpha_j}h_j$ and let μ be the number of vectors in a maximal linearly independent subset of $\{\alpha_1,\dots,\alpha_m\}$; say that $\alpha_1,\dots,\alpha_\mu$ are linearly independent. We then can define new holomorphic coordinates (still denoted by x) so that $\hat{f}_{j1}=x^{\alpha_j}$, $j=1,\dots,\mu$, see [23, p. 46] for details. Then we get

$$\begin{aligned}\bigwedge_1^m d\hat{f}_{j1} &= \bigwedge_1^\mu dx^{\alpha_j}\wedge\bigwedge_{\mu+1}^m(x^{\alpha_j}dh_j+h_jdx^{\alpha_j})\\ &= x^{\sum_{\mu+1}^m\alpha_j}\bigwedge_1^\mu dx^{\alpha_j}\wedge\bigwedge_{\mu+1}^m dh_j,\end{aligned} \tag{21}$$

where the last equality follows because $dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_\mu}\wedge dx^{\alpha_j}=0$, $\mu+1\le j\le m$, since $\alpha_1,\dots,\alpha_\mu,\alpha_j$ are linearly dependent. From the beginning we could also have assumed that $\varphi=\varphi_1\wedge\varphi_2$, where φ_1 is an anti-holomorphic $(n-\sum_1^q k_j+q-p)$-form and φ_2 is a (bundle valued) $(n,0)$-test form on Z. We now define

$$\Phi=\bigwedge_{\mu+1}^m\frac{d\bar{h}_j}{\bar{h}_j}\wedge\bigwedge_{m+1}^p\frac{\bar\partial\xi_j}{\xi_j}\wedge\bigwedge_1^q v^j\wedge\hat{\varphi}_1.$$

Using (21) we can now write (19) and (20) as

$$\pm\int_{\mathbb{C}^n_x}\frac{\prod_1^p\tilde{\chi}_j^\epsilon\prod_{p+1}^q\chi_j^\epsilon}{\prod_1^q\hat{f}_{j1}^{k_j}}\frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}}\wedge\cdots\wedge\frac{d\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}}\wedge\Phi\wedge\hat{\varphi}_2\rho, \tag{22}$$

$$\pm\lambda_1\cdots\lambda_p\int_{\mathbb{C}^n_x}\frac{\prod_1^q|\hat{f}_j|^{2\lambda_j}}{\prod_1^q\hat{f}_{j1}^{k_j}}\frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}}\wedge\cdots\wedge\frac{d\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}}\wedge\Phi\wedge\hat{\varphi}_2\rho. \tag{23}$$

Lemma 19. *Let $\mathcal{K}=\{i;\, x_i\,\big|\,x^{\alpha_j},\ \text{some } p+1\le j\le q\}$. For any fixed $r\in\mathbb{N}$, one can replace Φ in (22) and (23) by*

$$\Phi':=\Phi-\sum_{J\subseteq\mathcal{K}}(-1)^{|J|}\sum_{k_1,\dots,k_{|J|}=0}^{r+1}\frac{\partial^{|k|}\Phi}{\partial x_J^k}\bigg|_{x_J=0}\frac{x_J^k}{k!}$$

without affecting the integrals. Moreover, for any $I\subseteq\mathcal{K}$, we have that $\Phi'\wedge\Lambda_{i\in I}(d\bar{x}_i/\bar{x}_i)$ is C^r-smooth.

We replace Φ by Φ' in (22) and (23) and we write $d = d_{\mathcal{K}} + d_{\mathcal{K}^c}$, where $d_{\mathcal{K}}$ differentiates with respect to the variables x_i, $\bar{x}_i$ for $i \in \mathcal{K}$ and $d_{\mathcal{K}^c}$ differentiates with respect to the rest. Then we can write $(d\bar{x}^{\alpha_1}/\bar{x}^{\alpha_1}) \wedge \cdots \wedge (d\bar{x}^{\alpha_\mu}/\bar{x}^{\alpha_\mu}) \wedge \Phi'$ as a sum of terms, which we without loss of generality can assume to be of the form

$$\frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \frac{d_{\mathcal{K}}\bar{x}^{\alpha_{\nu+1}}}{\bar{x}^{\alpha_{\nu+1}}} \wedge \cdots \wedge \frac{d_{\mathcal{K}}\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi'$$
$$= \frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \Phi'' \wedge d\bar{x}_{\mathcal{K}},$$

where Φ'' is C^r-smooth and of bidegree $(0, n-\nu-|\mathcal{K}|)$ (possibly, $\Phi'' = 0$). Thus, (22) and (23) are finite sums of of integrals of the following type

$$\int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx, \tag{24}$$

$$\lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{\prod_1^q |\hat{f}_j|^{2\lambda_j}}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx, \tag{25}$$

where ψ is C^r-smooth and compactly supported.

We now first finish the proof of Proposition 18. First of all, it is well known that $\Gamma(\lambda)$ has a meromorphic continuation to $\mathbb{C}^q$. We have

$$\frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge d\bar{x}_{\mathcal{K}} = \sum_{\substack{|I|=\nu \\ I\subseteq \mathcal{K}^c}} C_I \frac{d\bar{x}_I}{\bar{x}_I} \wedge d\bar{x}_{\mathcal{K}}.$$

Let us assume that $I = \{1, \dots, \nu\} \subseteq \mathcal{K}^c$ and consider the contribution to (25) corresponding to this subset. This contribution equals

$$C_I \lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{|x^{\sum_1^q \lambda_j \alpha_j}|^2}{x^{\sum_1^q k_j \alpha_j}} \bigwedge_1^\nu \frac{d\bar{x}_j}{\bar{x}_j} \wedge \Psi(\lambda, x) \wedge d\bar{x}_{\mathcal{K}} \wedge dx \tag{26}$$
$$= \frac{C_I \prod_1^p \lambda_j}{\prod_{i=1}^\nu (\sum_1^q \lambda_j \alpha_{ji})} \int_{\mathbb{C}^n_x} \frac{\bigwedge_{i=1}^\nu \bar{\partial}|x_i|^{2\sum_1^q \lambda_j \alpha_{ji}} \prod_{i=\nu+1}^n |x_i|^{2\sum_1^q \lambda_j \alpha_{ji}}}{x^{\sum_1^q k_j \alpha_j}} \wedge$$
$$\wedge \Psi(\lambda, x) \wedge d\bar{x}_{\mathcal{K}} \wedge dx,$$

where $\Psi(\lambda, x) = \psi(x) \prod_1^q (\xi_j^{\lambda_j}/h_j^{k_j})$. It is well known (and not hard to prove, e.g., by integrations by parts as in [1], Lemma 2.1) that the *integral* on the right-hand side of (26) has an analytic continuation in λ to a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \geq 0\}$. (We thus choose r in Lemma 19 large enough so that we can integrate by parts.) If $p = 0$, then the coefficient in front of the integral is to be interpreted as 1 and Proposition 18 follows in this case. For $p > 0$, we see that the poles of (26), and consequently of $\Gamma(\lambda)$, in a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \geq 0\}$ are along hyperplanes of the form $0 = \sum_1^q \lambda_j \alpha_{ji}$, $1 \leq i \leq \nu$. But if $j > p$ and $i \leq \nu$, then $\alpha_{ji} = 0$ since $\{1, \ldots, \nu\} \subseteq \mathcal{K}^c = \{i;\, x_i \nmid x^{\alpha_j}, \forall j = p+1, \ldots, q\}$. Thus, the hyperplanes are of the form $0 = \sum_1^p \lambda_j \alpha_{ji}$ and Proposition 18 is proved except for the statement that at least for two j:s, the α_{ji} are non-zero. However, we see from (26) that if for some i we have $\alpha_{ji} = 0$ for all j but one, then the appearing λ_j in the denominator will be canceled by the numerator. Moreover, we may assume that the constant $C_I = \det(\alpha_{ji})_{1 \leq i,j \leq \nu}$ is non-zero which implies that we cannot have any λ_j^2 in the denominator.

We now prove Proposition 17. Consider (24). We have that $\alpha_1, \ldots, \alpha_\nu$ are linearly independent so we may assume that $A = (\alpha_{ij})_{1 \leq i,j \leq \nu}$ is invertible with inverse $B = (b_{ij})$. We make the non-holomorphic change of variables (13), where the "q" of (13) now should be understood as ν. Then we get $x^{\alpha_j} = y^{\alpha_j} \eta_j$, where $\eta_j > 0$ and smooth and $\eta_j^2 = 1/\xi_j$, $j = 1, \ldots, \nu$. Hence, $|\hat{f}_j|^2 = |y^{\alpha_j}|^2$, $j = 1, \ldots, \nu$. Expressed in the y-coordinates we get that $\Lambda_1^\nu (d\bar{x}^{\alpha_j}/\bar{x}^{\alpha_j}) \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx$ is a finite sum of terms of the form

$$\frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_{\nu'}}}{\bar{y}^{\alpha_{\nu'}}} \wedge \bar{y}_{\mathcal{K}'}\, d\bar{y}_{\mathcal{K}''} \wedge \psi_1, \tag{27}$$

where $\nu' \leq \nu$, ψ_1 is a C^r-smooth compactly supported form, and $\mathcal{K}'$ and $\mathcal{K}''$ are disjoint sets such that $\mathcal{K}' \cup \mathcal{K}'' = \mathcal{K}$. In order to give a contribution to (24) we see that ψ_1 must contain dy. In (27) we write $d = d_{\mathcal{K}} + d_{\mathcal{K}^c}$, and arguing as we did immediately after Lemma 19, (27) is a finite sum of terms of the form

$$\frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_{\nu''}}}{\bar{y}^{\alpha_{\nu''}}} \wedge \psi_2 \wedge d\bar{y}_{\mathcal{K}} \wedge dy,$$

where $\nu'' \leq \nu$ and ψ_2 is C^r-smooth and compactly supported. With abuse of notation we thus have that (24) is a finite sum of integrals of the form

$$\int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_\nu}}{\bar{y}^{\alpha_\nu}} \wedge \psi \wedge d\bar{y}_{\mathcal{K}} \wedge dy \tag{28}$$

$$= \int_{\mathbb{C}^n_x} \frac{\bigwedge_1^\nu d\chi_j^\epsilon \prod_{\nu+1}^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{y^{\sum_1^q k_j \alpha_j}} \wedge \Psi \wedge d\bar{y}_{\mathcal{K}} \wedge dy,$$

where Ψ is a C^r-smooth compactly supported $(n - |\mathcal{K}| - \nu)$-form; the equality follows since $\chi_j^\epsilon = \chi_j(|y^{\alpha_j}|^2/\epsilon_j)$, $j = 1, \ldots, \nu$. Now, (28) is essentially equal to equation (24) of [16] and the proof of Proposition 17 is concluded as in the proof of Proposition 8 in [16]. □

Proof of Lemma 19. The proof is similar to the proof of Lemma 9 in [16] but some modifications have to be done. First, it is easy to check by induction over $|\mathcal{K}|$ that $\Phi' \wedge \Lambda_{i\in I}(d\bar{x}_i/\bar{x}_i)$ is C^r-smooth for any $I \subseteq \mathcal{K}$; for $|\mathcal{K}| = 1$ this is just Taylor's formula for forms. It thus suffices to show that

$$d\bar{x}^{\alpha_1} \wedge \cdots \wedge d\bar{x}^{\alpha_\mu} \wedge \left.\frac{\partial^{|k|}\Phi}{\partial x_I^k}\right|_{x_I=0} = 0, \quad \forall I \subseteq \mathcal{K},\ k = (k_{i_1}, \ldots, k_{i_{|I|}}).$$

To show this, fix an $I \subseteq \mathcal{K}$ and let $L = \{j;\ x_i \nmid x^{\alpha_j}\ \forall i \in I\}$. Say for simplicity that

$$L = \{1, \ldots, \mu', \mu+1, \ldots, m', m+1, \ldots, p', p+1, \ldots, q'\},$$

where $\mu' \le \mu$, $m' \le m$, $p' \le p$, and $q' < q$. The fact that $q' < q$ follows from the definitions of $\mathcal{K}$, I, and L.

Consider, on the base variety Z, the smooth form

$$F = \bigwedge_1^{\mu'} d\bar{f}_{j1} \bigwedge_{\mu+1}^{m'} d\bar{f}_{j1} \bigwedge_{m+1}^{p'} (|f_{j1}|^2\bar\partial|f_j|^2 - \bar\partial|f_{j1}|^2|f_j|^2) \bigwedge_{j\in L} |f_j|^{2k_j} u^j_{k_j} \wedge \varphi_1.$$

It has bidegree $(0, n - \sum_{j\in L^c} k_j + q - q')$ so F has a vanishing pullback to $\cap_{j\in L^c}\{f_j = 0\}$ since this set has dimension $n - \sum_{j\in L^c} e_j < n - \sum_{j\in L^c} k_j + q - q'$ by our assumption about complete intersection. Thus, $\hat{F}$ has a vanishing pullback to $\{x_I = 0\} \subseteq \cap_{j\in L^c}\{\hat{f}_j = 0\}$. In fact, this argument shows that

$$\hat{F} = \sum \phi_j, \tag{29}$$

where the ϕ_j are smooth linearly independent forms such that each ϕ_j is divisible by $\bar{x}_i$ or $d\bar{x}_i$ for some $i \in I$. (It is the pull-back to $\{x_I = 0\}$ of the anti-holomorphic differentials of $\hat{F}$ that vanishes.) For the rest of the proof we let $\sum \phi_j$ denote such expressions and we note that they are invariant under holomorphic differential operators. Computing $\hat{F}$ we get

$$\hat{F} = \prod_{m+1}^{p'} |\hat{f}_{j1}|^4 \prod_{j\in L} \frac{|\hat{f}_j|^{2k_j}}{\hat{f}_{j1}^{k_j}} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d(\bar{x}^{\alpha_j}\bar{h}_j) \bigwedge_{m+1}^{p'} \bar\partial\xi_j \bigwedge_{j\in L} v^j \wedge \hat{\varphi}_1.$$

The "coefficient" $\prod_{m+1}^{p'} |\hat{f}_{j1}|^4 \prod_{j\in L}(|\hat{f}_j|^{2k_j}/\hat{f}_{j1}^{k_j})$ does not contain any $\bar{x}_i$ with $i \in I$ so we may divide (29) by it (recall that the ϕ_j are linearly independent) and we obtain

$$\begin{aligned}\sum \phi_j &= \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d(\bar{x}^{\alpha_j}\bar{h}_j) \bigwedge_{m+1}^{p'} \bar\partial \xi_j \bigwedge_{j\in L} v^j \wedge \hat{\varphi}_1 \\ &= \prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d\bar{h}_j \bigwedge_{m+1}^{p'} \bar\partial\xi_j \bigwedge_{j\in L} v^j \wedge \hat{\varphi}_1 \\ &\quad + \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \sum_{\mu+1}^{m'} d\bar{x}^{\alpha_j} \wedge \tau_j\end{aligned}$$

for some τ_j. We multiply this equality with

$$\bigwedge_{m'+1}^{m} d\bar{h}_j \bigwedge_{p'+1}^{p} \bar\partial\xi_j \bigwedge_{j\in L^c} v^j / \left(\prod_{\mu+1}^{m} \bar{h}_j \prod_{m+1}^{p} \xi_j \right)$$

and get

$$\prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \Phi + \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \sum_{\mu+1}^{m'} d\bar{x}^{\alpha_j} \wedge \tau_j = \sum \phi_j$$

for some new τ_j. We apply the operator $\partial^{|k|}/\partial x_I^k$ to this equality and then we pull back to $\{x_I = 0\}$, which makes the right-hand side vanish; (we construe however the result in $\mathbb{C}^n_x$). Finally, taking the exterior product with $\Lambda^{\mu}_{\mu'+1} d\bar{x}^{\alpha_j}$, which will make each term in under the summation sign on the left-hand side vanish, we arrive at

$$\prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu} d\bar{x}^{\alpha_j} \wedge \frac{\partial^{|k|}\Phi}{\partial x_I^k}\Big|_{x_I=0} = 0$$

and we are done. □

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