Noname manuscript No.
(will be inserted by the editor)

A weak space-time formulation for the linear stochastic heat equation

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Received: date / Accepted: date

Abstract We apply the well-known Banach-Nečas-Babuška inf-sup theory in a stochastic setting to introduce a weak space-time formulation of the linear stochastic heat equation with additive noise. We give sufficient conditions on the the data and on the covariance operator associated to the driving Wiener process, in order to have existence and uniqueness of the solution. We show the relation of the obtained solution to the so-called mild solution and to the variational solution of the same problem. The spatial regularity of the solution is also discussed. Finally, an extension to the case of linear multiplicative noise is presented.

Keywords Inf-sup theory · Stochastic linear heat equation · Additive noise · Linear multiplicative noise

Mathematics Subject Classification (2000) MSC 60H15 · MSC 35R60

1 Introduction

We consider a linear parabolic stochastic evolution problem of the form

$$\begin{aligned} &\mathrm{d}U(t) + A(t)U(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad t \in (0,T], \\ &U(0) = U_0. \end{aligned} \tag{1.1}$$

We assume that $A(t)$ is a random elliptic operator defined within a Gelfand triple setting as follows. Given separable Hilbert spaces V, H, we consider a Gelfand triple $V \subset H \subset V^*$, where V is continuously and densely embedded into H. We denote by $\langle \cdot, \cdot \rangle_H$ the inner product in H and by ${}_{V^*}\langle \cdot, \cdot \rangle_V$ the dual pairing

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between V and V^* with ${}_{V^*}\langle u, v\rangle_V = \langle u, v\rangle_H$, $\forall v \in V$ whenever $u \in H$. Further, we denote by $\mathscr{L}(H)$ the space of bounded linear operators on H and by $\mathscr{L}_2(H)$ the Hilbert-Schmidt operators.

Let $T \in (0, \infty)$ be fixed and let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space, with normal filtration $\Sigma = (\Sigma_t)_{t\in[0,T]}$. We assume that a progressively measurable map $A\colon [0,T] \times \Omega \times V \to V^*$, coercive and bounded $\mathrm{d}t \otimes \mathbb{P}$-a.s., is given, with associated bilinear form a given by $a(t, \omega; u, v) = {}_{V^*}\langle A(t,\omega)u, v\rangle_V$. We consider a predictable process with Bochner integrable trajectories $f \in L^2([0,T] \times \Omega; V^*)$ and we assume that $W = (W(t))_{t\in[0,T]}$ is a Q-Wiener process, with covariance operator $Q \in \mathscr{L}(H)$ of trace class, i.e., $Q^{\frac{1}{2}} \in \mathscr{L}_2(H,H) := \mathscr{L}_2(H)$.

In order to give a meaning to (1.1), we have to define what we mean by a solution. In the special case when A is independent of t and ω and considered as unbounded operator in H, we have the concepts of *weak* and *mild solution*, see [6].

Definition 1 (Weak and mild solution) Let the operator A be possibly unbounded, independent of ω and t and defined on a certain domain $D(A)$, i.e., $A\colon D(A) \subset H \to H$. A weak solution to (1.1) is an H-valued, predictable stochastic process $U(t)$, which is Bochner integrable $\mathbb{P}$-a.s. and satisfies

$$\begin{aligned}\langle U(t), v\rangle_H = \langle U_0, v\rangle_H - \int_0^t \langle U(s), A^* v\rangle_H \,\mathrm{d}s + \int_0^t \langle f(s), v\rangle_H \,\mathrm{d}s \\ + \int_0^t \langle \,\mathrm{d}W(s), v\rangle_H, \quad \mathbb{P}\text{-a.s.}, \forall v \in D(A^*),\ t \in [0,T].\end{aligned} \tag{1.2}$$

Moreover, if $-A$ is the generator a strongly continuous semigroup $(S(t))_{t\geq 0}$ in H and $\int_0^T \|S(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\mathrm{d}s < \infty$, then the unique weak solution coincides with the mild solution, given by the formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s) f(s) \,\mathrm{d}s + \int_0^t S(t-s) \,\mathrm{d}W(s), \quad t \in [0,T]. \tag{1.3}$$

We briefly recall how to switch from the Gelfand triple framework to the semigroup framework in Appendix A. Within the semigroup framework it is possible to prove results about about spatial regularity and temporal Hölder-continuity of the solution, by defining Sobolev spaces of fractional order, $\dot{H}^\beta := D(A^{\frac{\beta}{2}})$, and exploiting the semigroup theory. For example, in the parabolic case, when the semigroup is analytic, it was shown in [17] that if $U_0 \in L^2(\Omega; \dot{H}^\beta)$, $f = 0$, and $\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)} < \infty$ for some $\beta \geq 0$, then the mild solution satisfies

$$\|U(t)\|_{L^2(\Omega;\dot{H}^\beta)} \leq C\Big(\|U_0\|_{L^2(\Omega;\dot{H}^\beta)} + \|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}\Big), \quad t \in [0,T].$$

The concept of mild solution presents however the disadvantage of not being applicable whenever the operator does not generate a semigroup. This fact provides a good reason to look for more general concepts of solution that do not rely at all on such a theory.

The aim of this paper is to introduce a new concept of solution based on a weak formulation of the problem. In order to prove the mild solution formula (1.3), [6] proceeds from the space-time weak formulation (1.2) with time-independent deterministic test functions, to a weak formulation with time-dependent deterministic test functions,

$$\begin{aligned}\langle U(t), v(t)\rangle_H = \langle U_0, v(0)\rangle_H + \int_0^t \langle U(s), \dot{v}(s) - A^* v(s)\rangle_H \,\mathrm{d}s \\ + \int_0^t \langle f(s), v(s)\rangle_H \,\mathrm{d}s + \int_0^t \langle \,\mathrm{d}W(s), v(s)\rangle_H.\end{aligned}$$

This suggests the possibility of using a weak space-time formulation, which would be to find a pair (U_1, U_2) such that

$$\begin{aligned}&\int_0^t \langle U_1(s), -\dot{v}(s) + A^* v(s)\rangle_H \,\mathrm{d}s + \langle U_2, v(t)\rangle_H \\ &\quad = \langle U_0, v(0)\rangle_H + \int_0^t \langle f(s), v(s)\rangle_H \,\mathrm{d}s + \int_0^t \langle \,\mathrm{d}W(s), v(s)\rangle_H,\end{aligned}$$

for all v in a suitable class of test functions.

With a proper choice of function spaces, the well-posedness of this problem in the deterministic setting is obtained within the Banach-Nečas-Babuška inf-sup theory, see Section 2 below. In Section 3 we extend this to the stochastic evolution problem (1.1). The equation is solved ω-wise and the inf-sup theory allows to prove that a solution exists, is unique, and satisfies a bound that is expressed in terms of the data U_0, f, and W, $\mathbb{P}$-a.s. By taking the expectation of this, we achieve a standard estimate for the norm of the solution in the space $L^2([0,T]\times\Omega; V)\cap L^2(\Omega; \mathscr{C}([0,T]; H))$, which is consistent with standard estimates presented, for example, in [4, Chapt. 5]. In particular, under suitable assumptions, our solution coincides with the mild solution. In Section 5, we briefly discuss the spatial regularity under such assumptions.

A more general solution concept is the so-called *variational solution*, for which a comprehensive theory can be found, for example, in [12, Chapt. 4]. This theory applies to more general quasilinear equations, but we present it here for our linear equation.

Definition 2 (Variational solution) Assume that Q is trace class, i.e., $Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$. A continuous H-valued Σ-adapted process $(U(t))_{t\in[0,T]}$ is called a variational solution to (1.1), if for its $\mathrm{d}t\otimes\mathbb{P}$ equivalence class $\hat{U}$ we have $\hat{U} \in L^2([0,T]\times \Omega, \mathbb{P}\otimes \mathrm{d}t; V)$ and

$$U(t) = U_0 - \int_0^t A(s)\bar{U}(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s + \int_0^t \,\mathrm{d}W(s), \quad \mathbb{P}\text{-a.s.},$$

for any $t \in [0,T]$, where $\bar{U}$ is any V-valued progressively measurable $\mathrm{d}t\otimes\mathbb{P}$ version of $\hat{U}$.

We show that our solution coincides with such a solution, in particular, that our U_1 and U_2 play the roles of the $\bar{U}$ and U, respectively, in Definition 2.

Finally, the norm bound that we obtain for the solution operator of the linear problem with additive noise allows us to use a standard fixed point technique and extend our theory to the case of multiplicative noise. In Section 6 we present this in the case of linear multiplicative noise. This approach extends to semilinear equations under appropriate global Lipschitz assumptions.

Although our solution concept essentially is not more general the mild and variational solutions, the advantage is that it is characterized by a space-time weak formulation, which potentially can be used as the basis for numerical methods. This will be exploited in future work.

2 Preliminaries

2.1 The inf-sup theory

We recall the Banach-Nečas-Babuška (BNB) theorem, see [1,8], for example. Let V and W be Banach spaces, W reflexive, and consider a bounded bilinear form $\mathscr{B}\colon V \times W \to \mathbb{R}$, with

$$C_B := \sup_{0\neq w\in W} \sup_{0\neq v\in V} \frac{\mathscr{B}(v,w)}{\|v\|_V \|w\|_W} < \infty, \tag{BDD}$$

and the associated bounded linear operator $B\colon W \to V^*$, i.e., $B \in \mathscr{L}(W, V^*)$, defined by

$${}_V\langle v, Bw\rangle_{V^*} := \mathscr{B}(v,w), \quad \forall w \in W, \forall v \in V.$$

The operator B is boundedly invertible if and only if the following conditions are satisfied:

$$c_B := \inf_{0\neq w\in W} \sup_{0\neq v\in V} \frac{\mathscr{B}(v,w)}{\|v\|_V \|w\|_W} > 0, \tag{BNB1}$$

$$\forall v \in V, \quad \sup_{0\neq w\in W} \mathscr{B}(v,w) > 0. \tag{BNB2}$$

The constant c_B is called the inf-sup constant and, whenever both V and W are reflexive and (BNB1) holds, we have the identity

$$\inf_{0\neq w\in W} \sup_{0\neq v\in V} \frac{\mathscr{B}(v,w)}{\|v\|_V \|w\|_W} = \inf_{0\neq v\in V} \sup_{0\neq w\in W} \frac{\mathscr{B}(v,w)}{\|v\|_V \|w\|_W}, \tag{2.1}$$

which allows to swap the spaces where the infimum and the supremum are taken.

An immediate consequence of this is that the variational problem

given $F \in V^*$, find $w \in W : \mathscr{B}(v,w) = F(v), \quad \forall v \in V,$

i.e., solve $Bw = F$ in V^*, and its adjoint

given $G \in W^*$, find $v \in V : \mathscr{B}(v,w) = G(w), \quad \forall w \in W,$

i.e., solve $B^*v = G$ in W^*, are well-posed whenever (BDD), (BNB1) and (BNB2) hold. In particular, the well-posedness of the former is equivalent to the well-posedness of the latter and the norms of the respective solutions are bounded by

$$\|w\|_W \le \frac{1}{c_B}\|F\|_{V^*}, \quad \|v\|_V \le \frac{1}{c_B}\|G\|_{W^*}.$$

2.2 The inf-sup theory applied to an abstract parabolic problem

In recent years there has been a renewed interest for the tools presented above in order to deal with the linear heat equation starting from an abstract parabolic equation given in the Gelfand triple framework (see, for example, [2,3,5,13,14,15, 16]). Assume indeed that Hilbert spaces V, H are given, forming a Gelfand triple $V \subset H \subset V^*$ with bilinear forms

$$a(t\,;\cdot,\cdot)\colon V \times V \to \mathbb{R}, \quad t \in [0,T],$$

satisfying the following conditions for some positive numbers α, M_a:

$$\begin{aligned} |a(t;u,v)| &\le M_a \|u\|_V \|v\|_V, && t \in [0,T],\ u,v \in V,\\ a(t;v,v) &\ge \alpha \|v\|_V^2, && t \in [0,T],\ v \in V. \end{aligned}$$

For every $t \in [0,T]$, let $A(t)$ be the bounded linear operator from V to V^* associated with the bilinear form, i.e., $A(t) \in \mathcal{L}(V,V^*)$ and

$${}_{V^*}\langle A(t)u, v\rangle_V = a(t;u,v) = {}_V\langle u, A^*(t)v\rangle_{V^*}.$$

Consider now the problem

$$\begin{aligned} \dot{u}(t) + A(t)u(t) &= f(t) && \text{in } V^*,\ t \in (0,T),\\ u(0) &= u_0 && \text{in } H, \end{aligned} \tag{2.2}$$

where $\dot{u}(t)$ denotes the derivative of u with respect to t, i.e., $\dot{u}(t) := \frac{\mathrm{d}u}{\mathrm{d}t}$. Define the Lebesgue-Bochner spaces

$$\begin{aligned} \mathcal{Y} &= L^2([0,T];V),\\ \mathcal{X} &= L^2([0,T];V) \cap H^1((0,T);V^*), \end{aligned}$$

normed by

$$\begin{aligned} \|y\|_{\mathcal{Y}}^2 &= \|y\|_{L^2([0,T];V)}^2 = \int_0^T \|y(t)\|_V^2 \,\mathrm{d}t,\\ \|x\|_{\mathcal{X}}^2 &= \|x\|_{L^2([0,T];V)}^2 + \|\dot{x}\|_{L^2([0,T];V^*)}^2 = \int_0^T \|x(t)\|_V^2 + \|\dot{x}(t)\|_{V^*}^2 \,\mathrm{d}t. \end{aligned}$$

By the trace theorem, $\mathcal{X}$ is densely embedded in $\mathscr{C}([0,T];H)$ and there exists a constant M_e, uniform in the choice of V, such that

$$M_e := \sup_{0 \ne x \in \mathcal{X}} \frac{\|x(t)\|_{\mathscr{C}([0,T];H)}}{\|x\|_{\mathcal{X}}} < \infty.$$

Moreover, whenever $x, y \in \mathcal{X}$, integration by parts is possible:

$$\int_0^T \Big({}_{V^*}\langle \dot{x}(t), y(t)\rangle_V + {}_V\langle x(t), \dot{y}(t)\rangle_{V^*} \Big) \,\mathrm{d}t = \langle x(T), y(T)\rangle_H - \langle x(0), y(0)\rangle_H.$$

The reader can refer to [7] for a comprehensive presentation of these spaces.

A possible approach to solving the differential problem (2.2) is presented for example in [13] and it consists in integrating in time the dual pairing between the equation and a test function $y_1 \in \mathcal{Y}$ and taking the inner product between the initial condition and another test vector $y_2 \in H$, thus obtaining the following two equations:

$$\int_0^T \Big({}_{V^*}\langle \dot{u}(t), y_1(t)\rangle_V + a(t; u(t), y_1(t)) \Big)\, \mathrm{d}t = \int_0^T {}_{V^*}\langle f(t), y_1(t)\rangle_V \, \mathrm{d}t,$$
$$\langle u(0), y_2\rangle_H = \langle u_0, y_2\rangle_H.$$

Adding the equations and defining $\mathcal{Y}_H := \mathcal{Y} \times H$, Hilbert space normed by its product norm, gives the variational problem

$$u \in \mathcal{X} : \mathscr{B}(u, y) = \mathscr{F}(y), \quad \forall y = (y_1, y_2) \in \mathcal{Y}_H, \tag{2.3}$$

where the following bilinear and linear forms are used

$$\mathscr{B} \colon \mathcal{X} \times \mathcal{Y}_H \to \mathbb{R},$$
$$\mathscr{B}(x, y) := \int_0^T \Big({}_{V^*}\langle \dot{x}(t), y_1(t)\rangle_V + a(t; x(t), y_1(t)) \Big)\, \mathrm{d}t + \langle x(0), y_2\rangle_H,$$
$$\mathscr{F} \colon \mathcal{Y}_H \to \mathbb{R},$$
$$\mathscr{F}(y) := \int_0^T {}_{V^*}\langle f(t), y_1(t)\rangle_V \, \mathrm{d}t + \langle u_0, y_2\rangle_H.$$

We call this the *first* space-time variational formulation of (2.2). This is not the only way to include the initial condition in the variational formulation, but it will turn out to be the most suitable in our analysis.

Consider now the backward adjoint problem to (2.2):

$$\begin{aligned} -\dot{v}(t) + A^*(t)v(t) &= g(t) \quad &&\text{in } V^*,\ t \in (0, T),\\ v(T) &= v_T &&\text{in } H, \end{aligned} \tag{2.4}$$

whose *first* space-time variational formulation is given by

$$v \in \mathcal{X} : \mathscr{B}^*(y, v) = \mathscr{G}(y), \quad \forall y \in \mathcal{Y}_H. \tag{2.5}$$

Here the bilinear form is given by

$$\mathscr{B}^* \colon \mathcal{Y}_H \times \mathcal{X} \to \mathbb{R},$$
$$\mathscr{B}^*(y, x) := \int_0^T \Big({}_V\langle y_1(t), -\dot{x}(t)\rangle_{V^*} + a(t; y_1(t), x(t)) \Big)\, \mathrm{d}t + \langle y_2, x(T)\rangle_H,$$

and the load functional by

$$\mathscr{G} \colon \mathcal{Y}_H \to \mathbb{R},$$
$$\mathscr{G}(y) := \int_0^T {}_{V^*}\langle g(t), y_1(t)\rangle_V \, \mathrm{d}t + \langle y_2, v_T\rangle_H.$$

By defining a new load functional

$$\mathscr{F} \colon \mathcal{X} \to \mathbb{R},$$
$$\mathscr{F}(x) := \int_0^T {}_{V^*}\langle f(t), x(t)\rangle_V \, \mathrm{d}t + \langle u_0, x(0)\rangle_H,$$

and interchanging the roles of trial and test spaces, the *second* (or *weak*) space-time formulation of the original problem (2.2) is obtained:

$$u = (u_1, u_2) \in \mathcal{Y}_H : \mathscr{B}^*(u, x) = \mathscr{F}(x), \quad \forall x \in \mathcal{X}. \tag{2.6}$$

If a solution of (2.6) has the additional regularity $u_1 \in \mathcal{X}$, then an integration by parts shows that u_1 is a solution of the first problem (2.3) and that $u_2 = u_1(T)$. In this case the second component of the solution, u_2, can be understood as a continuous H-valued version of u_1, evaluated at time $t = T$. Therefore, u_2 is redundant and in other works the weak space-time formulation is

$$u \in \mathcal{Y} : \mathscr{B}^*(u, x) = \mathscr{F}(x), \quad \forall x \in \mathcal{X}_{0,\{T\}} := \{x \in \mathcal{X} : x(T) = 0\}.$$

However, in the present work we found it useful to keep u_2.

The first and the second formulations are related and the well-posedness of the former is equivalent to the well-posedness of the latter. More precisely, it holds that (by a suitable modification of the proofs in [13,15])

$$\begin{aligned} C_B &:= \sup_{0 \neq x \in \mathcal{X}} \sup_{0 \neq y \in \mathcal{Y}_H} \frac{\mathscr{B}^*(y, x)}{\|x\|_{\mathcal{X}} \|y\|_{\mathcal{Y}_H}} \leq \sqrt{2 \max\{1, M_a^2\} + M_e^2}, \\ c_B &:= \inf_{0 \neq x \in \mathcal{X}} \sup_{0 \neq y \in \mathcal{Y}_H} \frac{\mathscr{B}^*(y, x)}{\|x\|_{\mathcal{X}} \|y\|_{\mathcal{Y}_H}} \geq \frac{\alpha \min\{M_a^{-2}, 1\}}{\sqrt{2 \max\{\alpha^{-2}, 1\} + M_e^2}}, \end{aligned} \tag{2.7}$$

and that for any $y \in \mathcal{Y}_H$ the following condition is satisfied:

$$\sup_{0 \neq x \in \mathcal{X}} \mathscr{B}^*(y, x) \geq \min\{1, \alpha\} \|y\|^2_{\mathcal{Y}_H}.$$

This shows that the operator $B^* \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_H^*)$, associated with the bilinear form $\mathscr{B}^*(\cdot, \cdot)$ via $\mathscr{B}^*(y, x) = {}_{\mathcal{Y}_H}\langle y, B^* x \rangle_{\mathcal{Y}_H^*}$ is boundedly invertible. This, in turn, implies that the operator $B \in \mathcal{L}(\mathcal{Y}_H, \mathcal{X}^*)$ associated with $\mathscr{B}^*(\cdot, \cdot)$ via $\mathscr{B}^*(y, x) = {}_{\mathcal{X}^*}\langle By, x \rangle_{\mathcal{X}}$ is also boundedly invertible, with the same inf-sup constant, see (2.1). Moreover, for $f \in L^2([0, T]; V^*)$ and $u_0 \in H$, we have $\mathscr{F} \in \mathcal{X}^*$. Hence, (2.6) is well-posed.

3 A weak space-time formulation of the stochastic problem

In order to introduce the weak space-time formulation for the equation (1.1) we will follow the idea outlined in Subsection 2.2. We consider spaces $\mathcal{X}$ and $\mathcal{Y}$ restricted to a time interval $[0, t]$, for fixed $t \in [0, T]$, endowed with their respective natural norms. We denote these spaces by

$$\begin{aligned} \mathcal{Y}_0^t &= L^2([0, t]; V), \\ \mathcal{X}_0^t &= L^2([0, t]; V) \cap H_0^1((0, t); V^*), \end{aligned}$$

with the convention that $\mathcal{X} = \mathcal{X}_0^T$ and $\mathcal{Y} = \mathcal{Y}_0^T$.

We assume that the family of operators $A(s, \omega)$ is as in Section 1, i.e., that its bilinear forms satisfy the following conditions for some positive numbers α, M_a:

$$\begin{aligned} |a(s, \omega; u, v)| &\leq M_a \|u\|_V \|v\|_V, && s \in [0, T],\ \omega \in \Omega, \quad u, v \in V, \\ a(s, \omega; v, v) &\geq \alpha \|v\|_V^2, && s \in [0, T],\ \omega \in \Omega, \quad v \in V. \end{aligned}$$

We introduce a family of problems parametrized by (t,ω), defined by the bilinear forms

$$
\begin{aligned}
&\mathscr{B}^*_{t,\omega}\colon (\mathcal{Y}^t_0 \times H) \times \mathcal{X}^t_0 \to \mathbb{R},\\
&\mathscr{B}^*_{t,\omega}(y,x) := \int_0^t \Big({}_{V^*}\langle y_1(s), -\dot{x}(s)\rangle_V + a(s,\omega; y_1(s), x(s))\Big)\,\mathrm{d}s\\
&\qquad\qquad + \langle y_2, x(t)\rangle_H,
\end{aligned}
$$

and the load functionals

$$
\begin{aligned}
&\mathscr{W}_{t,\omega}\colon \mathcal{X}^t_0 \to \mathbb{R},\\
&\mathscr{W}_{t,\omega}(x) := \mathscr{F}_{t,\omega}(x) + \widetilde{\mathscr{W}_{t,\omega}}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{F}_{t,\omega}(x) &= \int_0^t {}_{V^*}\langle f(s,\omega), x(s)\rangle_V \,\mathrm{d}s + \langle U_0(\omega), x(0)\rangle_H,\\
\widetilde{\mathscr{W}_{t,\omega}}(x) &= \Big(\int_0^t \langle \mathrm{d}W(s), x(s)\rangle_H\Big)(\omega).
\end{aligned}
$$

The weak space-time formulation reads, for almost every $(t,\omega) \in [0,T]\times \Omega$:

$$
U(t,\omega) \in (\mathcal{Y}^t_0 \times H) : \mathscr{B}^*_{t,\omega}(U(t,\omega), x) = \mathscr{W}_{t,\omega}(x), \quad \forall x \in \mathcal{X}^t_0. \tag{3.1}
$$

Since our assumption on $a(s,\omega;\cdot,\cdot)$ is uniform with respect to s,ω with constants α, M_a, we conclude that the bilinear forms $\mathscr{B}^*_{t,\omega}$ satisfy the inf-sup conditions uniformly in t,ω with the same constants C_B, c_B as in (2.7). This means that for almost every $(t,\omega) \in [0,T]\times\Omega$, the operator $B_{t,\omega} \in \mathcal{L}(\mathcal{Y}^t_0 \times H, (\mathcal{X}^t_0)^*)$ associated to $\mathscr{B}^*_{t,\omega}(\cdot,\cdot)$ via $\mathscr{B}^*_{t,\omega}(y,x) = {}_{(\mathcal{X}^t_0)^*}\langle B_{t,\omega} y, x\rangle_{\mathcal{X}^t_0}$ is boundedly invertible. Moreover, the norm of the inverse operator $B^{-1}_{t,\omega}$ is bounded by c_B^{-1}, uniformly in t,ω.

Focusing now on the right-hand side, we assume that $f(\cdot,\omega) \in \mathcal{Y}^*$ and that $U_0(\omega) \in H$. Then, for $x \in \mathcal{X}^t_0$, it holds that

$$
\begin{aligned}
|\mathfrak{F}_{t,\omega}(x)| &= \Big|\int_0^t {}_{V^*}\langle f(s,\omega), x(s)\rangle_V \,\mathrm{d}s + \langle U_0(\omega), x(0)\rangle_H\Big|\\
&\le \Big(\int_0^t \|f(s,\omega)\|^2_{V^*}\,\mathrm{d}s\Big)^{\frac12}\Big(\int_0^t \|x(s)\|_V{}^2\,\mathrm{d}s\Big)^{\frac12} + \|U_0(\omega)\|_H\|x(0)\|_H\\
&\lesssim \|f(\cdot,\omega)\|_{(\mathcal{Y}^t_0)^*}\|x\|_{\mathcal{Y}^t_0} + \|U_0(\omega)\|_H\|x\|_{\mathcal{X}^t_0}\\
&\le \big(\|f(\cdot,\omega)\|_{(\mathcal{Y}^t_0)^*} + \|U_0(\omega)\|_H\big)\|x\|_{\mathcal{X}^t_0},
\end{aligned}
$$

showing that $\mathfrak{F}_{t,\omega} \in (\mathcal{X}^t_0)^*$ for almost every $(t,\omega)\in[0,T]\times\Omega$. In particular, by monotonicity in t, it follows that

$$
\mathbb{E}\Big[\sup_{t\in[0,T]} \|\mathfrak{F}_{t,\cdot}\|_{(\mathcal{X}^t_0)^*}\Big] \lesssim \mathbb{E}\Big[\|f\|_{\mathcal{Y}^*} + \|U_0\|_H\Big]. \tag{3.2}
$$

The final step is provided by the following lemma, which shows that $\widetilde{\mathscr{W}_{t,\omega}} \in (\mathcal{X}^t_0)^*$ with an estimate similar to the one in (3.2).

Lemma 1 *If $Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$, then there exists a process $K \in L^2(\Omega;\mathscr{C}([0,T];\mathbb{R}))$ such that, for almost every $(t,\omega) \in [0,T] \times \Omega$,*

$$\|\widetilde{\mathscr{W}}_{t,\omega}\|_{(\mathcal{X}_0^t)^*} \lesssim K(t,\omega) \tag{3.3}$$

and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} K(t,\omega)^2\Big] \lesssim \mathbb{E}\Big[\int_0^T \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}t\Big] = T\|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}. \tag{3.4}$$

Hence, $\widetilde{\mathscr{W}}_{t,\omega} \in (\mathcal{X}_0^t)^*$ for almost every $(t,\omega) \in [0,T] \times \Omega$ and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\widetilde{\mathscr{W}}_{t,\omega}\|^2_{(\mathcal{X}_0^t)^*}\Big] \lesssim \mathbb{E}\Big[\int_0^T \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}t\Big] = T\|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}.$$

Proof We use an analytic semigroup $(S_0(t))_{t\geq 0}$ generated by a fixed operator $-A_0$. This can be achieved, for example, by freezing the coefficients and setting $A_0^* := A^*(t_0,\omega_0)$, for a fixed pair (t_0,ω_0). Then $-A_0^*$ is the generator of an analytic semigroup $(S_0^*(t))_{t\geq 0}$, and we consider the adjoint problem (2.4) on $[0,t]$, with $A^*(\cdot)$ replaced by A_0^*. Problem (2.4) is hence uniquely solvable, which means that the operator $B_0^* \colon \mathcal{X}_0^t \to (\mathcal{Y}_0^t \times H)^*$ associated to the bilinear form in (2.5) is a bijection. We can hence write any $x \in \mathcal{X}_0^t$ as $B_0^{-*}B_0^*x$, which, in view of the semigroup theory, can be represented as:

$$x(s) = (B_0^{-*}B_0^*x)(s) = \int_s^t S_0^*(r-s)(-\dot{x}(r) + A_0^*x(r))\,\mathrm{d}r + S_0^*(t-s)x(t).$$

We insert this expression into the weak stochastic integral to get

$$\begin{aligned}
&\int_0^t \langle \mathrm{d}W(s), x(s)\rangle_H \\
&\quad= \int_0^t \int_s^t \langle \mathrm{d}W(s), S_0^*(r-s)(-\dot{x}(r) + A_0^*x(r))\rangle_H\,\mathrm{d}r \\
&\qquad+ \int_0^t \langle \mathrm{d}W(s), S_0^*(t-s)x(t)\rangle_H \\
&\quad= \int_0^t \Big\langle \int_0^r S_0(r-s)\mathrm{d}W(s), (-\dot{x}(r) + A_0^*x(r))\Big\rangle_H\,\mathrm{d}r \\
&\qquad+ \Big\langle \int_0^t S_0(t-s)\mathrm{d}W(s), x(t)\Big\rangle_H.
\end{aligned}$$

Here we used the stochastic Fubini theorem and $(S_0(t))^* = S_0^*(t)$. It follows that

$$\begin{aligned}
&\Big|\int_0^t \langle \mathrm{d}W(s), x(s)\rangle_H\Big| \\
&\quad\leq \Big(\int_0^t \Big\|\int_0^r S_0(r-s)\mathrm{d}W(s)\Big\|_V^2\,\mathrm{d}r\Big)^{\frac{1}{2}}\Big(\int_0^t \|-\dot{x}(r) + A_0^*x(r)\|^2_{V^*}\,\mathrm{d}r\Big)^{\frac{1}{2}} \\
&\qquad+ \Big\|\int_0^t S_0(t-s)\mathrm{d}W(s)\Big\|_H \|x(t)\|_H \\
&\quad\lesssim \Big(\int_0^t \Big\|\int_0^r S_0(r-s)\,\mathrm{d}W(s)\Big\|_V^2\,\mathrm{d}r + \Big\|\int_0^t S_0(t-s)\,\mathrm{d}W(s)\Big\|_H^2\Big)^{\frac{1}{2}} \|x\|_{\mathcal{X}_0^t}
\end{aligned}$$

This implies (3.3) with

$$K(t,\cdot) := \Big(\int_0^t \Big\| \int_0^r S_0(r-s)\,\mathrm{d}W(s) \Big\|_V^2 \,\mathrm{d}r + \Big\| \int_0^t S_0(t-s)\,\mathrm{d}W(s) \Big\|_H^2 \Big)^{\frac{1}{2}}.$$

By monotonicity in t, we have

$$\sup_{t\in[0,T]} K(t,\cdot)^2 \lesssim \int_0^T \Big\| \int_0^r S_0(r-s)\mathrm{d}W(s) \Big\|_V^2 \,\mathrm{d}r + \sup_{t\in[0,T]} \Big\| \int_0^t S_0(t-s)\mathrm{d}W(s) \Big\|_H^2.$$

By taking the expectation it follows that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} K(t,\cdot)^2 \Big] \lesssim \mathbb{E}\Big[\int_0^T \Big\| \int_0^r S_0(r-s)\mathrm{d}W(s) \Big\|_V^2 \,\mathrm{d}r + \sup_{t\in[0,T]} \Big\| \int_0^t S_0(t-s)\mathrm{d}W(s) \Big\|_H^2 \Big].$$

By means of Ito's isometry, Doob's maximal inequality, and by using the smoothing property of the semigroup, it follows that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S_0(t-s)\,\mathrm{d}W(s) \Big\|_H^2 \Big] \le 16\mathbb{E}\Big[\int_0^T \|Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \,\mathrm{d}s \Big]$$

and

$$\mathbb{E}\Big[\int_0^T \Big\| \int_0^r S_0(r-s)\,\mathrm{d}W(s) \Big\|_V^2 \,\mathrm{d}r \Big] \le \frac{1}{2}\mathbb{E}\Big[\int_0^T \|Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \,\mathrm{d}s \Big].$$

A proof of these statements is found in [4, Chapt. 3, Lemma 5.2]. This proves (3.4).

Putting together the results presented above, and recalling that the second component of the solution is a H-valued version of the first component, we can rewrite everything in a compact form in the following theorem:

Theorem 1 (Existence and uniqueness) *If $U_0 \in L^2(\Omega; H)$, $f \in L^2(\Omega \times [0,T]; V^*)$ and $Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$, then there exists a unique solution $U \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; \mathscr{C}([0,T]; H))$ to the problem (3.1). Its norm satisfies the bound*

$$\mathbb{E}\Big[\int_0^T \|U_1(s)\|_V^2 \,\mathrm{d}s + \sup_{t\in[0,T]} \|U_2(t)\|_H^2 \Big] \lesssim c_B^{-1} \Big(\mathbb{E}\Big[\int_0^T \|f(s)\|_{V^*}^2 \,\mathrm{d}s + \|U_0\|_H^2 \Big] + \|Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \Big),$$

where the constant hidden in $\lesssim$ is purely a numerical factor.

Proof In view of the preliminary results about the ω-wise invertibility of the operator $B_{t,\omega}$, and according to the remarks about $\mathscr{F}_{t,\omega}$ and Lemma 1, we have that for fixed ω and for any $t \in [0,T]$, there exists a unique solution to the problem (3.1), which satisfies the bound

$$\int_0^t \|U_1(s,\omega)\|_V^2 \,\mathrm{d}s + \|U_2(t,\omega)\|_H^2 \lesssim c_B^{-1} \Big(K(t,\omega) + \|f(\omega)\|_{(\mathcal{Y}_0^t)^*}^2 + \|U_0(\omega)\|_H^2 \Big).$$

Since the bound holds for every t, we can in particular take the expectation of the supremum over $t \in [0,T]$ at both sides, thus obtaining by monotonicity with respect to t that

$$\mathbb{E}\Big[\int_0^T \|U_1(s)\|_V^2 \,\mathrm{d}s + \sup_{t\in[0,T]} \|U_2(t)\|_H^2\Big] \\ \lesssim c_B^{-1}\mathbb{E}\Big[\|f\|_{\mathcal{Y}^*}^2 + \|U_0\|_H^2 + \sup_{t\in[0,T]} K(t,\cdot)^2\Big],$$

which, in view of Lemma 1, becomes

$$\mathbb{E}\Big[\int_0^T \|U_1(s)\|_V^2 \,\mathrm{d}s + \sup_{t\in[0,T]} \|U_2(t)\|_H^2\Big] \\ \lesssim c_B^{-1}\mathbb{E}\Big[\|f\|_{\mathcal{Y}^*}^2 + \|U_0\|_H^2 + \|Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\Big].$$

This, together with the remark in Section 2.2 that U_2 is a version of U_1, concludes the proof of the theorem.

Notice that the assumption on $Q^{\frac{1}{2}}$ is the same as in the definition of the other concepts of solutions presented and our result is consistent with the analogous results obtained in [4, Chapt. 5].

In order to prepare for the treatment of multiplicative noise in Section 6, we consider also the modified problem

$$\mathrm{d}U(t) + A(t)U(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + \Psi(t)\,\mathrm{d}W(t), \quad t \in (0,T], \\ U(0) = U_0,$$

where Ψ is a predictable operator-valued process, $\Psi \in L^2([0,T]\times\Omega;\mathscr{L}(H))$. We use the same weak formulation (3.1) but with the modified load functional

$$\widetilde{\mathscr{W}}_{t,\omega}(x) = \Big(\int_0^t \langle \Psi(s)\,\mathrm{d}W(s), x(s)\rangle_H\Big)(\omega).$$

A straightforward modification of the proof of Lemma 1 leads to the following.

Lemma 2 *If* $\Psi Q^{\frac{1}{2}} \in L^2([0,T]\times\Omega;\mathscr{L}_2(H))$, *then there exists a process* K *such that, for almost every* $(t,\omega) \in [0,T]\times\Omega$,

$$\|\widetilde{\mathscr{W}}_{t,\omega}\|_{(\mathcal{X}_0^t)^*} \le K(t,\omega).$$

Moreover,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\widetilde{\mathscr{W}}_{t,\omega}\|_{(\mathcal{X}_0^t)^*}^2\Big] \lesssim \mathbb{E}\Big[\int_0^T \|\Psi(t)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\,\mathrm{d}t\Big].$$

Remark 1 An alternative approach that does not involve the semi-group theory while estimating the stochastic integral ω-wise is based on a modification of the right-hand side of the equation (3.1), by noticing that it is equal to:

$$\widetilde{\mathscr{W}}_{t,\omega}(x) = \int_0^t {}_{V^*}\langle W(s,\omega), -\dot{x}(s)\rangle_V + \langle W(t,\omega), x(t)\rangle_H.$$

This approach presents however the drawback of requiring the process defining the noise to have the same spatial regularity as the solution that we are looking for. Indeed, by assuming that the Wiener process W belongs to a smaller space, namely that

$$W \in L^2(\Omega; \mathscr{C}([0,T]; H) \cap L^2([0,T]; V)),$$

we can obtain that, for any $t \in [0,T]$,

$$\|\widetilde{\mathscr{W}_{t,\omega}}\|_{(\mathcal{X}_0^t)^*} \lesssim \|W\|_{L^2([0,t];V)} + \|W(t)\|_H.$$

We can now use such an estimates in the same fashion as above, to finally obtain the following result.

Theorem 2 (Existence and uniqueness) *If* $U_0 \in L^2(\Omega; H)$, $f \in L^2(\Omega \times [0,T]; V^*)$ *and* $Q^{\frac{1}{2}} \in \mathscr{L}_2(H,V)$, *then there exists a unique solution* $U \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; \mathscr{C}([0,T]; H))$ *to the problem* (3.1). *Its norm satisfies the bound*

$$\mathbb{E}\Big[\int_0^T \|U_1(s)\|_V^2 \,\mathrm{d}s + \sup_{t\in[0,T]} \|U_2(t)\|_H^2\Big] \lesssim c_B^{-1}\Big(\mathbb{E}\Big[\int_0^T \|f(s)\|_{V^*}^2 \,\mathrm{d}s + \|U_0\|_H^2\Big] + \|Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H,V)}^2\Big),$$

where the constant hidden in $\lesssim$ *is purely a numerical factor.*

4 Properties of the solution

We want now to find a connection between the solution presented here and the other concepts of solution previously introduced. To this end, an important role is played by the second component U_2 of the solution $U = (U_1, U_2)$, that we keep in our ω-wise formulation, by choosing test functions in $\mathcal{X}$ and not in $\mathcal{X}_{0,\{T\}}$.

In the following sections we will omit to write explicitly the ω-dependence of the solution and of the data, whenever this does not lead to ambiguities.

4.1 Connection with the variational solution

In order to show the connection between our solution and the variational solution, we choose test functions $x \in \mathcal{X}_0^t$, for any $t \in [0,T]$, that are constant in time, i.e., $x(s) \equiv \xi \in V$, $s \in [0,t]$. Hence, for any $\xi \in V$, it holds almost surely that

$$\int_0^t {}_{V^*}\langle A(s)U_1(s), \xi\rangle_V \,\mathrm{d}s + \langle U_2(t), \xi\rangle_H = \int_0^t {}_{V^*}\langle f(s), \xi\rangle_V \,\mathrm{d}s + \langle U_0, \xi\rangle_H + \int_0^t \langle \,\mathrm{d}W(s), \xi\rangle_H,$$

which can be rewritten as

$${}_{V^*}\Big\langle U_2(t) - \Big(U_0 - \int_0^t A(s)U_1(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s + \int_0^t \mathrm{d}W(s)\Big), \xi\Big\rangle_V = 0,$$

which, in turn, implies

$$U_2(t) \overset{V^*}{=} U_0 - \int_0^t A(s)U_1(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s + \int_0^t \mathrm{d}W(s), \quad \mathbb{P}\text{-a.s..}$$

With the representation above, we can now rely on the theory in [12] to claim that such an identity holds not only in V^* but in H and that our solution is a variational solution.

4.2 Connection with the mild solution

Assume in the following that $-A$ is independent of t and ω and hence generates an analytic semigroup $(S(t))_{t\geq 0}$. Then the following theorem holds:

Theorem 3 *Let U be the mild solution* (1.3) *to the problem* (1.1) *and assume that $(U_1, U_2(t)) \in \mathcal{Y}_0^t \times H$ is the weak space-time solution to the same problem. Then, for any $t \in [0,T]$, $U_1 \overset{\mathcal{Y}}{=} U$ and $U_2(t) \overset{H}{=} U(t)$.*

Proof For any $t \in [0,T]$ and for any $x \in \mathcal{X}_0^t$, we have $\mathbb{P}$-a.s. that

$$\begin{aligned}\int_0^t ({}_V\langle U_1(s), -\dot{x}(s) + A^*x(s)\rangle_{V^*})\,\mathrm{d}s + \langle U_2(t), x(t)\rangle_H \\ = \int_0^t {}_{V^*}\langle f(s), x(s)\rangle_V\,\mathrm{d}s + \langle U_0, x(0)\rangle_H + \int_0^t \langle\,\mathrm{d}W(s), x(s)\rangle_H.\end{aligned} \tag{4.1}$$

We choose now test functions $x = v$, where $v \in \mathcal{X}_0^t$ is the solution to the deterministic backward equation (2.4) over the time interval $[0,t]$, with arbitrary final data ξ_t and load function g. Its variational formulation is given by (2.5), that is

$$\begin{aligned}\int_0^t {}_V\langle y_1(s), -\dot{v}(s) + A^*v(s)\rangle_{V^*}\,\mathrm{d}s + \langle y_2, v(t)\rangle_H \\ = \int_0^t {}_V\langle y_1(s), g(s)\rangle_{V^*}\,\mathrm{d}s + \langle y_2, \xi_t\rangle_H,\end{aligned} \tag{4.2}$$

for all $y \in \mathcal{Y}_0^t \times H$. The solution to such a problem is given by the mild formula

$$v(s) = S^*(t-s)\xi_t + \int_s^t S^*(r-s)g(r)\,\mathrm{d}r, \quad s \in [0,t], \tag{4.3}$$

where S^* is the semigroup generated by $-A^*$, namely $S^*(s) = \mathrm{e}^{-sA^*}$. By substituting $x = v$ in (4.1) and $y = (U_1, U_2(t))$ in (4.2), we obtain

$$\begin{aligned}\int_0^t {}_V\langle U_1(s), g(s)\rangle_{V^*}\,\mathrm{d}s + \langle U_2(t), \xi_t\rangle_H \\ = \int_0^t {}_{V^*}\langle f(s), v(s)\rangle_V\,\mathrm{d}s + \langle U_0, v(0)\rangle_H + \int_0^t \langle\,\mathrm{d}W(s), v(s)\rangle_H,\end{aligned}$$

which, by (4.3) is in turn equal to

$$\begin{aligned}
&= \int_0^t {}_{V^*}\langle f(s), S^*(t-s)\xi_t\rangle_V \,\mathrm{d}s + \int_0^t {}_{V^*}\Big\langle f(s), \int_s^t S^*(r-s)g(r)\,\mathrm{d}r\Big\rangle_V \,\mathrm{d}s \\
&\quad + \langle U_0, S^*(t)\xi_t\rangle_H + \Big\langle U_0, \int_s^t S^*(r-s)g(r)\,\mathrm{d}r\Big\rangle_H \\
&\quad + \int_0^t \langle\, \mathrm{d}W(s), S^*(t-s)\xi_t\rangle_H + \int_0^t \Big\langle\, \mathrm{d}W(s), \int_s^t S^*(r-s)g(r)\,\mathrm{d}r\Big\rangle_H .
\end{aligned}$$

By manipulating the dual pairings in a suitable way, changing the order of integration (using the stochastic version of Fubini's theorem), and using the mild solution formula (1.3), we get

$$\begin{aligned}
&\int_0^t {}_V\langle U_1(s), g(s)\rangle_{V^*}\,\mathrm{d}s + \langle U_2(t), \xi_t\rangle_H \\
&\quad = \Big\langle S(t)U_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}W(s), \xi_t\Big\rangle_H \\
&\qquad + \int_0^t {}_V\Big\langle S(s)U_0 + \int_0^s S(s-r)f(r)\,\mathrm{d}r + \int_0^s S(s-r)\,\mathrm{d}W(r), g(s)\Big\rangle_{V^*}\,\mathrm{d}s, \\
&\quad = \langle U(t), \xi_t\rangle_H + \int_0^t {}_V\langle U(s), g(s)\rangle_{V^*}\,\mathrm{d}s,
\end{aligned}$$

which reads

$$ {}_{\mathcal{Y}_0^t}\langle U_1 - U, g\rangle_{(\mathcal{Y}_0^t)^*} + \langle U_2(t) - U(t), \xi_t\rangle_H = 0. $$

Since (g, ξ_t) is arbitrary in $L^2([0,t];V^*)\times H$, and $t\in[0,T]$, it follows that

$$ U_1 \overset{\mathcal{Y}}{=} U, \qquad\qquad U_2(t) \overset{H}{=} U(t), \quad t\in[0,T],\ \mathbb{P}\text{-a.s.} $$

Remark 2 This is consistent with the fact that U_1 is a V-valued version of U_2 and that U_2 is a continuous H-valued function of time. Moreover, in the same way as in [4,6,9] we can also derive a connection to the weak solution (1.2). We omit the details.

5 Regularity

As already pointed out, it is possible to switch from the Gelfand triple setting to the semigroup setting whenever A is independent of t and ω. Moreover, the semigroup $S(t) = \mathrm{e}^{-tA}$ is analytic and fractional powers of A are well defined. Define $\dot H^\beta$ as the domain of $A^{\frac{\beta}{2}}$ and consider the spaces

$$\begin{aligned}
\mathcal{Y}_0^{t,\beta} &:= L^2([0,t];\dot H^{\beta+1}), \\
\mathcal{X}_0^{t,\beta} &:= L^2([0,t];\dot H^{1-\beta}) \cap H^1((0,t);\dot H^{-1-\beta}),
\end{aligned}$$

normed by

$$\begin{aligned}
\|y\|^2_{\mathcal{Y}_0^{t,\beta}} &:= \int_0^t \|A^{\frac{\beta+1}{2}} y(s)\|_H^2\,\mathrm{d}s, \\
\|x\|^2_{\mathcal{X}_0^{t,\beta}} &:= \int_0^t \big(\|A^{\frac{1-\beta}{2}} x(s)\|_H^2 + \|A^{-\frac{1+\beta}{2}} \dot x(s)\|_H^2\big)\,\mathrm{d}s.
\end{aligned}$$

The spaces in the previous sections correspond to $\beta = 0$. In particular, as before, we use the notation $\mathcal{Y}^\beta = \mathcal{Y}_0^{T,\beta}$ and $\mathcal{X}^\beta = \mathcal{X}_0^{T,\beta}$. The space $\mathcal{Y}_0^{t,\beta} \times \dot{H}^\beta$ endowed with its product norm $\|\cdot\|_{\mathcal{Y}_0^{t,\beta}\times\dot{H}^\beta}$ and the space $\mathcal{X}_0^{t,\beta}$ endowed with the $\|\cdot\|_{\mathcal{X}_0^{t,\beta}}$-norm are Hilbert spaces.

There is a dense embedding $\mathcal{X}^\beta \hookrightarrow \mathscr{C}([0,T];\dot{H}^{-\beta})$, i.e., for any $x \in \mathcal{X}^\beta$,

$$\|A^{-\frac{\beta}{2}}x\|_{\mathscr{C}([0,T];\dot{H}^{-\beta})} \lesssim \|x\|_{\mathcal{X}^\beta},$$

where, again, the underlying constant is uniform in the choice of V. A proof of this fact can be found in [7,10], and relies on the properties of the interpolating space

$$(\dot{H}^{1-\beta}, \dot{H}^{-1-\beta})_{\frac{1}{2}} = \dot{H}^{-\beta}.$$

We introduce a new bilinear form, $\mathscr{B}^*_{t,\omega,\beta}$, given by the original one, $\mathscr{B}^*_{t,\omega}$, restricted to the newly introduced domains, that is

$$\mathscr{B}^*_{t,\omega,\beta} \colon (\mathcal{Y}_0^{t,\beta} \times \dot{H}^\beta) \times \mathcal{X}_0^{t,\beta} \to \mathbb{R},$$

together with a new load functional,

$$\begin{aligned}&\mathscr{W}_{t,\omega,\beta} \colon \mathcal{X}_0^{t,\beta} \to \mathbb{R},\\ &\mathscr{W}_{t,\omega,\beta}(x) := \mathscr{F}_{t,\omega,\beta}(x) + \widetilde{\mathscr{W}}_{t,\omega,\beta}(x),\end{aligned}$$

given by $\mathscr{F}_{t,\omega}$ and $\widetilde{\mathscr{W}}_{t,\omega}$ defined on the new domains introduced above.

The weak space-time formulation reads hence, for almost every ω and $t \in [0,T]$:

$$U_\beta(t,\omega) \in \mathcal{Y}_0^{t,\beta} \times \dot{H}^\beta \colon \mathscr{B}^*_{t,\omega,\beta}(U_\beta(t,\omega), x) = \mathscr{W}_{t,\omega,\beta}(x), \quad \forall x \in \mathcal{X}_0^{t,\beta}. \tag{5.1}$$

It is possible to prove that the conditions (BDD), (BNB1) and (BNB2) still hold, with the same constants C_B and c_B as before. The proof of this follows from a straightforward modification of the proof for the deterministic framework in [13] or [15], taking in account the remarks made for its extension to the stochastic framework in Section 3. It will therefore be omitted.

In the following lemma we give sufficient conditions on the load functional $\mathscr{W}_{t,\omega,\beta}$ in order to have a unique solution.

Lemma 3 *With the notation introduced above, the following facts hold true:*

- *If* $f \in \mathcal{Y}_0^{t,\beta-2}$ *and* $U_0 \in \dot{H}^\beta$, $\mathbb{P}$*-a.s., then* $\mathscr{F}_{t,\omega,\beta} \in (\mathcal{X}_0^{t,\beta})^*$, $\mathbb{P}$*-a.s. Moreover, if* $f \in L^2(\Omega;\mathcal{Y}^{\beta-2})$ *and* $U_0 \in L^2(\Omega;\dot{H}^\beta)$*, then*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\mathscr{F}_{t,\omega,\beta}\|^2_{(\mathcal{X}_0^{t,\beta})^*}\Big] \lesssim \mathbb{E}\Big[\|f\|^2_{\mathcal{Y}_H^{\beta-2}} + \|U_0\|^2_{\dot{H}^\beta}\Big].$$

- *If* $Q^{\frac{1}{2}} \in \mathscr{L}_2(H,\dot{H}^\beta)$*, then* $\widetilde{W}_{t,\omega,\beta} \in (\mathcal{X}_0^{t,\beta})^*$, $\mathbb{P}$*-a.s. Moreover,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\widetilde{W}_{t,\omega,\beta}\|^2_{(\mathcal{X}_0^{t,\beta})^*}\Big] \lesssim \int_0^T \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H,\dot{H}^\beta)}\,\mathrm{d}t.$$

Proof The first statement is obvious. In order to prove the second one, one can use the same notation and techniques as in Section 3, together with the employment of the following properties to derive an analogous to Lemma 1:

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|A^{\frac{\beta}{2}} \int_0^t S(t-s)\mathrm{d}W(s)\|_H^2\Big] \lesssim \int_0^T \|A^{\frac{\beta}{2}} Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}s$$

and

$$\mathbb{E}\Big[\int_0^T \|A^{\frac{\beta-1}{2}} \int_0^r S(r-s)\mathrm{d}W(s)\|_H^2\,\mathrm{d}r\Big] \lesssim \int_0^T \|A^{\frac{\beta}{2}} Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}s.$$

These two properties follows from a direct generalizations of the ones presented in [4, Lemma 5.2].

The previous lemma, together with the initial remarks about the fulfilment of the conditions (BDD), (BNB1), and (BNB2), gives the following result.

Theorem 4 *Let $\beta \geq 0$ and $f \in L^2(\Omega;\mathcal{Y}^{\beta-2})$, $U_0 \in L^2(\Omega;\dot{H}^\beta)$, and $Q^{\frac{1}{2}} \in \mathscr{L}_2(H,\dot{H}^\beta)$. Then the problem (5.1) has a unique solution $U \in L^2(\Omega;\mathcal{Y}^\beta) \cap L^2(\Omega;\mathscr{C}([0,T];\dot{H}^\beta))$ and its norm is bounded by*

$$\mathbb{E}\Big[\int_0^T \|U_1(s)\|^2_{\dot{H}^{\beta+1}}\,\mathrm{d}s + \sup_{t\in[0,T]} \|U_2(t)\|^2_{\dot{H}^\beta}\Big]$$
$$\lesssim c_B^{-1}\mathbb{E}\Big[\int_0^T \|f(s)\|^2_{\dot{H}^{\beta-1}}\,\mathrm{d}s + \int_0^T \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H,\dot{H}^\beta)}\,\mathrm{d}s + \|U_0\|^2_{\dot{H}^\beta}\Big].$$

6 Linear multiplicative noise

In this section we use the theory developed in the previous sections to prove existence and uniqueness to the weak space-time solution to the problem

$$\begin{aligned} &\mathrm{d}U(t) + A(t)U(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + (B(t)U(t))\,\mathrm{d}W(t), \quad t \in (0,T], \\ &U(0) = U_0. \end{aligned} \tag{6.1}$$

Here $B(t,\omega) \in \mathscr{L}(H,\mathscr{L}(H))$, with further assumptions on its (t,ω)-dependence to be specified below. As we have done before, we introduce an ω-wise weak formulation. In order to do so we introduce a new load functional $\mathscr{W}^v_{t,\omega}$ defined by

$$\begin{aligned} &\mathscr{W}^v_{t,\omega}\colon \mathcal{X}_0^t \to \mathbb{R}, \\ &\mathscr{W}^v_{t,\omega}(x) := \mathscr{F}_{t,\omega}(x) + \widetilde{\mathscr{W}^v_{t,\omega}}(x), \end{aligned}$$

where

$$\widetilde{\mathscr{W}^v_{t,\omega}}(x) = \Big(\int_0^t \langle (B(s)\,v(s))\,\mathrm{d}W(s), x(s)\rangle_H\Big)(\omega),$$

for $(t,\omega) \in [0,T]\times\Omega$ and $v \in \mathscr{S}_T := L^2(\Omega;L^2([0,T];V)) \cap L^2(\Omega;\mathscr{C}([0,T];H))$. The weak space-time formulation of problem (6.1) reads hence, for almost every $(t,\omega) \in [0,T]\times\Omega$,

$$U(t,\omega) \in \mathcal{Y}_0^t{\times}_H : \mathscr{B}^*_{t,\omega}(U(t,\omega),x) = \mathscr{W}^U_{t,\omega}(x), \quad \forall x \in \mathcal{X}_0^t. \tag{6.2}$$

We use Banach's fixed point theorem for the linear operator $\mathscr{T}\colon v \mapsto U$ that maps $v \in \mathscr{S}_T$ to the solution of of the problem

$$U(t,\omega) \in \mathcal{Y}_0^t \times H : \mathscr{B}^*_{t,\omega}(U(t,\omega), x) = \mathscr{W}^v_{t,\omega}(x), \quad \forall x \in \mathcal{X}_0^t. \tag{6.3}$$

We will show that $\mathscr{T}\colon \mathscr{S}_T \to \mathscr{S}_T$ is a contraction, if T is small.

We make the further assumption that B is predictable, uniformly bounded with respect to ω, and L^p in time for some $p > 2$, i.e., for some constant κ,

$$\Big(\int_0^T \|B(t,\omega)\|^p_{\mathscr{L}(H,\mathscr{L}(H))} \,\mathrm{d}t \Big)^{1/p} \le \kappa, \quad \mathbb{P}\text{-a.s.} \tag{6.4}$$

We may then prove the following lemma.

Lemma 4 *For any $v \in \mathscr{S}_T$ and B as in* (6.4)*, it holds that*

$$\mathbb{E}\Big[\int_0^T \|(B(s,\cdot)\, v(s,\cdot))Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\mathrm{d}s \Big] \lesssim T^{\frac{p}{p-2}} \,\|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\|v\|^2_{\mathscr{S}_T}.$$

Proof We use Hölder's inequality to get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \|B(s,\cdot)\, v(s,\cdot))Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\mathrm{d}s \Big] \\
&\quad \le \mathbb{E}\Big[\int_0^T \|B(s,\cdot)\|^2_{\mathscr{L}(H,\mathscr{L}(H))} \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \|v(s,\cdot)\|^2_H \,\mathrm{d}s \Big] \\
&\quad \le \mathbb{E}\Big[\sup_{t\in[0,T]} \|v(s,\cdot)\|^2_H \Big(\int_0^T \|B(s,\cdot)\|^2_{\mathscr{L}(H,\mathscr{L}(H))} \,\mathrm{d}s \Big) \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \Big] \\
&\quad \le \mathbb{E}\Big[\sup_{t\in[0,T]} \|v(s,\cdot)\|^2_H \, T^{\frac{p}{p-2}} \Big(\int_0^T \|B(s,\cdot)\|^p_{\mathscr{L}(H,\mathscr{L}(H))} \,\mathrm{d}s \Big)^{\frac{2}{p}} \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \Big] \\
&\quad \le T^{\frac{p}{p-2}} \kappa^2 \,\|v\|^2_{\mathscr{S}_T} \|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)},
\end{aligned}$$

where in the last line we used (6.4).

By combining Lemma 4 with Lemma 2, with $\Psi = Bv$, we see that $\widetilde{\mathscr{W}}^v_{t,\omega} \in (\mathcal{X}_0^t)^*$ and

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]} \|\widetilde{\mathscr{W}}^v_{t,\omega}\|^2_{(\mathcal{X}_0^t)^*} \Big] &\lesssim \mathbb{E}\Big[\int_0^T \|(B(t)v(t))Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\mathrm{d}t \Big] \\
&\lesssim T^{\frac{p}{p-2}} \,\|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\|v\|^2_{\mathscr{S}_T}.
\end{aligned}$$

If $U_0 \in L^2(\Omega; H)$, $f \in L^2(\Omega \times [0,T]; V^*)$, $Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$, then we may argue as in Theorem 1 to conclude that (6.3) has a unique solution with

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \|U_1(s)\|^2_V \,\mathrm{d}s + \sup_{t\in[0,T]} \|U_2(t)\|^2_H \Big] \\
&\qquad \lesssim \mathbb{E}\Big[\int_0^T \|f(s)\|^2_{V^*} \,\mathrm{d}s + \|U_0\|^2_H \Big] + T^{\frac{p}{p-2}} \,\|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\|v\|^2_{\mathscr{S}_T}.
\end{aligned}$$

Hence, the solution operator $\mathscr{T}$ maps $\mathscr{S}_T$ to itself. An application of the previous bound with $f = 0$, $U_0 = 0$ shows that it is a contraction, if T is small. We thus have a unique solution for T small and since the interval of existence does not depend on the size of the data f, U_0, we may extend it to a global solution.

We summarize the result in the following theorem:

Theorem 5 (Existence and uniqueness) *If* $U_0 \in L^2(\Omega; H)$, $f \in L^2(\Omega \times [0,T]; V^*)$, $Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$, *and* $B \in L^\infty(\Omega; L^p([0,T]; \mathscr{L}(H, \mathscr{L}(H))))$ *for some* $p > 2$, *then there exists a unique solution* $U \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; \mathscr{C}([0,T]; H))$ *to the problem* (6.2)

Remark 3 This approach extends easily to a semilinear equation of the form

$$\mathrm{d}U(t) + A(t)U(t)\,\mathrm{d}t = F(t, U(t))\,\mathrm{d}t + B(t, U(t))\,\mathrm{d}W(t)$$

under appropriate global Lipschitz assumptions on the nonlinear operators F, B.

Remark 4 Under the hypotheses of Section 5, and by assuming that for some $p > 2$ the following bound holds uniformly in ω,

$$\Big(\int_0^T \|B(t,\omega)\|^p_{\mathscr{L}(\dot{H}^\beta, \mathscr{L}(\dot{H}^\beta))}\Big)^{1/p} \mathrm{d}t \le \kappa,$$

we may extend the results of Theorem 4 to the case of linear multiplicative noise.

A Connection between the semigroup framework and the variational framework

In this appendix we outline how to switch between the Gelfand triple and semigroup frameworks. This is based on [7, Chapt. XVIII.3], [12, Appendix F], [9], and [5].

Recall that

$$V \overset{J}{\hookrightarrow} H \overset{\Phi}{\cong} H^* \overset{J^*}{\hookrightarrow} V^*$$

where J and J^* are dense embeddings and Φ is the Riesz isomorphism. We want to modify the operator A introduced above, under the hypothesis that it is deterministic and time-independent, so that it becomes an unbounded operator $\tilde{A}$ from H into H. Define

$$D(A) \subset H = \{v \in V : Av \in J^*\Phi(H)\},$$

and the new operator $\tilde{A}$ by

$$\begin{aligned} \tilde{A}\colon D(\tilde{A}) &\subset H \to H, \\ D(\tilde{A}) &:= J(D(A)), \\ \tilde{A} &:= \Phi^{-1}(J^*)^{-1}AJ^{-1}. \end{aligned}$$

$\tilde{A}$ is thus an unbounded densely defined linear operator, positive definite because of the coercivity of the bilinear form. If the bilinear form $a(\cdot,\cdot)$ associated to A is symmetric and J is a compact embedding, then $\tilde{A}$ is self-adjoint, boundedly invertible, with compact inverse $\tilde{A}^{-1} := JA^{-1}J^*\Phi$, and this implies that we can use the spectral theorem in order to define the semigroup and fractional powers of $\tilde{A}$. Alternatively, we can argue that such an operator is the generator of a strongly continuous semigroup of contractions and such a semigroup is holomorphic, as outlined in [11, Theorem 1.52], and it is hence possible to define fractional powers of $\tilde{A}$. In order to simplify the notation, we finally omit the embeddings and denote $\tilde{A}$ by A.

For the other way around, i.e., how to switch from the semigroup framework to the Gelfand triple framework, we refer to [12, Appendix F, Remark F.0.6].

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